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**Publisher Citation**  

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Sasakian metric as a Ricci soliton
and related results

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Abstract: We prove the following results: (i) A Sasakian metric as a non-trivial Ricci soliton is null $\eta$-Einstein, and expanding. Such a characterization permits to identify the Sasakian metric on the Heisenberg group $\mathcal{H}^{2n+1}$ as an explicit example of (non-trivial) Ricci soliton of such type. (ii) If an $\eta$-Einstein contact metric manifold $M$ has a vector field $V$ leaving the structure tensor and the scalar curvature invariant, then either $V$ is an infinitesimal automorphism, or $M$ is $D$-homothetically fixed $K$-contact.

MSC: 53C15, 53C25, 53D10

Keywords: Ricci soliton, Sasakian metric, Null $\eta$-Einstein, $D$-homothetically fixed $K$-contact structure, Heisenberg group.

1 Introduction

A Ricci soliton is a natural generalization of an Einstein metric, and is defined on a Riemannian manifold $(M, g)$ by

$$(\mathcal{L}_{V} g)(X,Y) + 2 \text{Ric}(X,Y) + 2 \lambda g(X,Y) = 0$$

where $\mathcal{L}_{V} g$ denotes the Lie derivative of $g$ along a vector field $V$, $\lambda$ a constant, and arbitrary vector fields $X, Y$ on $M$. The Ricci soliton is said to be shrinking, steady, and expanding accordingly as $\lambda$ is negative, zero, and positive respectively. Actually, a Ricci soliton is a generalized fixed point of Hamilton's Ricci flow [7]: $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. For
details, see Chow et al. [4]. The vector field $V$ generates the Ricci soliton viewed as a special solution of the Ricci flow. A Ricci soliton is said to be a gradient Ricci soliton, if $V = -\nabla f$ (up to a Killing vector field) for a smooth function $f$. Ricci solitons are also of interest to physicists who refer to them as quasi-Einstein metrics (for example, see Friedan [6]).

An odd dimensional analogue of Kaehler geometry is the Sasakian geometry. The Kaehler cone over a Sasakian Einstein manifold is a Calabi-Yau manifold which has application in physics in superstring theory based on a 10-dimensional manifold that is the product of the 4-dimensional space-time and a 6-dimensional Ricci-flat Kaehler (Calabi-Yau) manifold (see Candelas et al. [3]). Sasakian geometry has been extensively studied since its recently perceived relevance in string theory. Sasakian Einstein metrics have received a lot of attention in physics, for example, $p$-brane solutions in superstring theory, Maldacena conjecture (AdS/CFS duality) [9]. For details, see Boyer, Galicki and Matzeu [2].

In [12] Sharma showed that if a $K$-contact (in particular, Sasakian) metric is a gradient Ricci soliton, then it is Einstein. This was also shown later independently by He and Zhu [8] for the Sasakian case. Recently, Sharma and Ghosh [13] proved that a 3-dimensional Sasakian metric which is a non-trivial (i.e. non-Einstein) Ricci soliton, is homothetic to the standard Sasakian metric on $\text{nil}^3$. In this paper, we generalize these results and also answer the following question of H.-D. Cao (cited in [8]): “Does there exist a shrinking Ricci soliton on a Sasakian manifold, which is not Einstein?” by proving

**Theorem 1** If the metric of a $(2n + 1)$-dimensional Sasakian manifold $M (\eta, \xi, g, \varphi)$ is a non-trivial (non-Einstein) Ricci soliton, then (i) $M$ is null $\eta$-Einstein (i.e. $D$-homothetically fixed and transverse Calabi-Yau), (ii) the Ricci soliton is expanding, and (iii) the generating vector field $V$ leaves the structure tensor $\varphi$ invariant, and is an infinitesimal contact $D$-homothetic transformation.

Conversely, we consider the following question: “What can we say about an $\eta$-Einstein contact metric manifold $M$ which admits a vector field $V$ that leaves $\varphi$ invariant?” and answer it by assuming the invariance of the scalar curvature under $V$, in the form of the following result.

**Theorem 2** If an $\eta$-Einstein contact metric manifold $M$ admits a vector field $V$ that leaves the structure tensor $\varphi$ and the scalar curvature invariant,
then either $V$ is an infinitesimal automorphism, or $M$ is $D$-homothetically fixed and $K$-contact.

**Remark 1** Note that a Ricci soliton as a Sasakian metric is different from the Sasaki-Ricci soliton in the context of transverse Kaehler structure in a Sasakian manifold, for example see Futaki et al. [5]).

**Remark 2** Boyer et al. [2] have studied $\eta$-Einstein geometry as a class of distinguished Riemannian metrics on contact metric manifolds, and proved the existence of $\eta$-Einstein metrics on many different compact manifolds. We would also like to point out that Zhang [18] showed that compact Sasakian manifolds with constant scalar curvature and satisfying certain positive curvature condition is $\eta$-Einstein.

**Remark 3** Theorem 2 provides a generalization of the infinitesimal version of the following result of Tanno [15] “The group of all diffeomorphisms $\Phi$ which leave the structure tensor $\varphi$ of a contact metric manifold $M$ invariant, is a Lie transformation group, and coincides with the automorphism group $A$ if $M$ is Einstein.” Note that the scalar curvature of an Einstein metric is constant. We also note that the set of all vector fields on a contact metric manifold $M$, that leave $\varphi$ and scalar curvature invariant, forms a Lie subalgebra of the Lie algebra of all smooth vector fields on $M$.

## 2 A Brief Review Of Contact Geometry

A $(2n + 1)$-dimensional smooth manifold is said to be contact if it has a global 1-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ on $M$. For a contact 1-form $\eta$ there exists a unique vector field $\xi$ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, we obtain a Riemannian metric $g$ and a $(1,1)$-tensor field $\varphi$ such that

$$d\eta(X, Y) = g(X, \varphi Y), \eta(X) = g(X, \xi), \varphi^2 = -I + \eta \otimes \xi$$  \hspace{1cm} (2)

$g$ is called an associated metric of $\eta$ and $(\varphi, \eta, \xi, g)$ a contact metric structure. Following [1] we recall two self-adjoint operators $h = \frac{1}{2} \mathcal{L}_\xi \varphi$ and $l = R(., \xi)\xi$. The tensors $h, h\varphi$ are trace-free and $h\varphi = -\varphi h$. We also have these formulas for a contact metric manifold.

$$\nabla_X \xi = -\varphi X - \varphi h X$$  \hspace{1cm} (3)
\[ l - \varphi l \varphi = -2(h^2 + \varphi^2) \]  \hspace{1cm} (4)

\[ \nabla_\xi h = \varphi - \varphi l - \varphi h^2 \]  \hspace{1cm} (5)

\[ Tr l = Ric(\xi, \xi) = 2n - Tr h^2 \]  \hspace{1cm} (6)

where \( \nabla, R, Ric \) and \( Q \) denote respectively, the Riemannian connection, curvature tensor, Ricci tensor and Ricci operator of \( g \). For details see [1]

A vector field \( V \) on a contact metric manifold \( M \) is said to be an infinitesimal contact transformation if \( \mathcal{L}_V \eta = \sigma \eta \) for some smooth function \( \sigma \) on \( M \). \( V \) is said to be an infinitesimal automorphism of the contact metric structure if it leaves all the structure tensors \( \eta, \xi, g, \varphi \) invariant (see Tanno [14]).

A contact metric structure is said to be \( K \)-contact if \( \xi \) is Killing with respect to \( g \), equivalently, \( h = 0 \). The contact metric structure on \( M \) is said to be Sasakian if the almost Kaehler structure on the cone manifold \((M \times \mathbb{R}^+, r^2 g + dr^2)\) over \( M \), is Kaehler. Sasakian manifolds are \( K \)-contact and \( K \)-contact 3-manifolds are Sasakian. For a Sasakian manifold,

\[ (\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \]  \hspace{1cm} (7)

\[ R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad Q\xi = 2n\xi \]  \hspace{1cm} (8)

For a Sasakian manifold, the restriction of \( \varphi \) to the contact sub-bundle \( D \) \( (\eta = 0) \) is denoted by \( J \) and \((D, J, d\eta)\) defines a Kaehler metric on \( D \), with the transverse Kaehler metric \( g^T \) related to the Sasakian metric \( g \) as \( g = g^T + \eta \otimes \eta \). One finds by a direct computation that the transverse Ricci tensor \( Ric^T \) of \( g^T \) is given by

\[ Ric^T(X, Y) = Ric(X, Y) + 2g(X, Y) \]

for arbitrary vector fields \( X, Y \) in \( D \). The Ricci form \( \rho \) and transverse Ricci form \( \rho^T \) are defined by

\[ \rho(X, Y) = Ric(X, \varphi Y), \quad \rho^T(X, Y) = Ric^T(X, \varphi Y) \]

for \( X, Y \in D \). The basic first Chern class \( 2\pi c_1^B \) of \( D \) is represented by \( \rho^T \). In case \( c_1^B = 0 \), the Sasakian structure is said to be null (transverse Calabi-Yau). We refer to [2] for details.
A contact metric manifold $M$ is said to be $\eta$-Einstein in the wider sense, if the Ricci tensor can be written as

$$Ric(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)$$

(9)

for some smooth functions $\alpha$ and $\beta$ on $M$. It is well-known (Yano and Kon [17]) that $\alpha$ and $\beta$ are constant if $M$ is $K$-contact, and has dimension greater than 3.

Given a contact metric structure $(\eta, \xi, g, \varphi)$, let $\bar{\eta} = a\eta, \bar{\xi} = \frac{1}{a}\xi, \bar{\varphi} = \varphi, \bar{g} = ag + a(a - 1)\eta \otimes \eta$ for a positive constant $a$. Then $(\bar{\eta}, \bar{\xi}, \bar{\varphi}, \bar{g})$ is again a contact metric structure. Such a change of structure is called a $D$-homothetic deformation, and preserves many basic properties like being $K$-contact (in particular, Sasakian). It is straightforward to verify that, under a $D$-homothetic deformation, a $K$-contact $\eta$-Einstein manifold transforms to a $K$-contact $\eta$-Einstein manifold such that $\bar{\alpha} = \frac{\alpha + 2 - 2a}{a}$ and $\bar{\beta} = 2n - \bar{\alpha}$. We remark here that the particular value: $\alpha = -2$ remains fixed under a $D$-homothetic deformation, and as $\alpha + \beta = 2n$, $\beta$ also remains fixed. Thus, we state the following definition.

Definition 1 A $K$-contact $\eta$-Einstein manifold with $\alpha = -2$ is said to be $D$-homothetically fixed.

3 Proofs Of The Results

Proof Of Theorem 1: Using the Ricci soliton equation (1) in the commutation formula (Yano [16], p.23)

$$\mathcal{L}_V \nabla g - \nabla \mathcal{L}_V g - \nabla_{[V,X]} g)(Y, Z) =$$

$$-g((\mathcal{L}_V \nabla)(X, Y), Z) - g((\mathcal{L}_V \nabla)(X, Z), Y)$$

(10)

we derive

$$g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z Ric)(X, Y)$$

$$- (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z)$$

(11)

As $\xi$ is Killing, we have $\mathcal{L}_\xi Ric = 0$ which, in view of (3), the last equation of (8) and $h = 0$, is equivalent to $\nabla_\xi Q = Q \varphi - \varphi Q$. But for a Sasakian
manifold, $Q$ commutes with $\varphi$, and hence $\text{Ric}$ is parallel along $\xi$. Moreover, differentiating the last equation of (8), we have $(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X$. Substituting $\xi$ for $Y$ in (11) and using these consequences we obtain

$$(\mathcal{L}_V \nabla)(X, \xi) = -2Q\varphi X + 4n\varphi X \quad (12)$$

Differentiating this along an arbitrary vector field $Y$, using (7) and the last equation of (8), we find

$$(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) - (\mathcal{L}_V \nabla)(X, \varphi Y) = -2(\nabla_Y Q)\varphi X + 2\eta(X)QY - 4n\eta(X)Y$$

The use of the foregoing equation in the commutation formula [16]:

$$(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z) \quad (13)$$

for a Riemannian manifold, shows that

$$(\mathcal{L}_V R)(X, Y)\xi - (\mathcal{L}_V \nabla)(Y, \varphi X) = -2(\nabla_X Q)\varphi Y + 2(\nabla_Y Q)\varphi X + 2\eta(Y)QX - 2\eta(X)QY + 4n\eta(Y)X \quad (14)$$

Equation (1) gives $(\mathcal{L}_V g)(X, \xi) + 2(2n + \lambda)\eta(X) = 0$, which in turn, gives

$$(\mathcal{L}_V \eta)(X) - g(\mathcal{L}_V \xi, X) + 2(\lambda + 2n)\eta(X) = 0 \quad (15)$$

$$\eta(\mathcal{L}_V \xi) = (2n + \lambda) \quad (16)$$

where we used the Lie-derivative of $g(\xi, \xi) = 1$ along $V$. Next, Lie-differentiating the formula $R(X, \xi)\xi = X - \eta(X)\xi$ [a consequence of the first formula in (8)] along $V$, and using equations (14) and (16) provides

$$4(QX - 2nX) - g(\mathcal{L}_V \xi, X)\xi + 2(2n + \lambda)X = -((\mathcal{L}_V \eta)(X))\xi$$

By the direct application of (15) to the the above equation we find

$$\text{Ric}(X, Y) = (n - \frac{\lambda}{2})g(X, Y) + (n + \frac{\lambda}{2})\eta(X)\eta(Y) \quad (17)$$
which shows that $M$ is $\eta$-Einstein with scalar curvature

$$r = 2n(n + 1) - n\lambda$$

(18)

At this point, we recall the following integrability formula [12]:

$$\mathcal{L}_V r = -\Delta r + 2\lambda r + 2|Q|^2$$

(19)

for a Ricci soliton, where $\Delta r = -\text{div} Dr$. A straightforward computation using (17) gives the squared norm of the Ricci operator as $|Q|^2 = 2n(n^2 - n\lambda + \frac{\lambda^2}{4} + 4n^2)$. Using this and (18) in (19), we obtain the quadratic equation $(2n + \lambda)(2n + 4 - \lambda) = 0$. As $\lambda = -2n$ corresponds to $g$ becoming Einstein, we must have $\lambda = 2n + 4$ and hence the soliton is expanding, which proves part (ii). Moreover, equation (18) reduces to $r = -2n$. Thus equation (17) assumes the form

$$\text{Ric}(Y, Z) = -2g(Y, Z) + 2(n + 1)\eta(Y)\eta(Z)$$

(20)

Hence, as defined in Section 2, $M$ is a $D$-homothetically fixed null $\eta$-Einstein manifold, proving part (i). Using (20) in (11) provides

$$(\mathcal{L}_V \nabla)(Y, Z) = 4(n + 1)\{\eta(Y)\phi Z + \eta(Z)\phi Y\}$$

(21)

Differentiating this along $X$, using equations (3) and (7), incorporating the resulting equation in (13), and finally contracting at $X$ we get

$$(\mathcal{L}_V \text{Ric})(Y, Z) = 8(n + 1)\{g(Y, Z) - (2n + 1)\eta(Y)\eta(Z)\}$$

(22)

Equation (20) reduces the soliton equation (1) to the form

$$(\mathcal{L}_V g)(Y, Z) = -4(n + 1)\{g(Y, Z) + \eta(Y)\eta(Z)\}$$

(23)

Next, Lie-differentiating (20) along $V$, and using (23) shows

$$(\mathcal{L}_V \text{Ric})(Y, Z) = 8(n + 1)\{g(Y, Z) + \eta(Y)\eta(Z)\}$$

$$+ 2(n + 1)\{\eta(Z)(\mathcal{L}_V \eta)(Y) + \eta(Y)(\mathcal{L}_V \eta)Z\}$$

(24)

Comparing equations (22) with (24) and substituting $\xi$ for $Z$ leads to

$$\mathcal{L}_V \eta = -4(n + 1)\eta$$

(25)
Therefore, substituting $\xi$ for $Z$ in (23) and using (25) we immediately get $\mathcal{L}_V \xi = 4(n + 1)\xi$. Operating (25) by $d$, noting $d$ commutes with $\mathcal{L}_V$ and using the first equation of (2) we find

$$(\mathcal{L}_V d\eta)(X,Y) = -4(n + 1)g(X,\varphi Y)$$

Its comparison with the Lie-derivative of the first equation of (2) and the use of (23) yields $\mathcal{L}_V \varphi = 0$, completing the proof.

Before proving Theorem 2, we state and prove the following lemma.

**Lemma 1** If a vector field $V$ leaves the structure tensor $\varphi$ of the contact metric manifold $M$ invariant, then there exists a constant $c$ such that

(i) $\mathcal{L}_V \eta = c\eta$,  
(ii) $\mathcal{L}_V \xi = -c\xi$,  
(iii) $\mathcal{L}_V g = c(g + \eta \otimes \eta)$.

Though this lemma was proved by Mizusawa in [10], to make the paper self-contained, we provide a slightly different proof as follows.

**Proof:** Lie-differentiating the formulas $\varphi \xi = 0$ and $\eta(\varphi X) = 0$ and using $\mathcal{L}_V \varphi = 0$, we find $\mathcal{L}_V \xi = -c\xi$, and $\mathcal{L}_V \eta = c\eta$ for a smooth function $c$ on $M$. Next, Lie-derivative of the formula $\eta(X) = g(X, \xi)$ along $V$ gives

$$(\mathcal{L}_V g)(X, \xi) = 2c\eta(X)$$  \hspace{1cm} (26)

The Lie-derivative of the first equation of (2) along $V$ provides

$$(\mathcal{L}_V g)(X, \varphi Y) = ((dc) \wedge \eta)(X,Y) + cg(X, \varphi Y)$$  \hspace{1cm} (27)

Substituting $\xi$ for $Y$ in the above equation we get $dc = (\xi c)\eta$. Taking its exterior derivative, and then exterior product with $\eta$ shows $(\xi c)(d\eta) \wedge \eta = 0$. By definition of the contact structure, $(d\eta) \wedge \eta$ is nowhere zero on $M$, and so $\xi c = 0$. Hence $dc = 0$, i.e. $c$ is constant. Using this consequence, and equations (26) and (27) we obtain (iii), completing the proof.

**Proof Of Theorem 2:** By virtue of Lemma 1, we have

$$(\mathcal{L}_V g)(Y, Z) = c\{g(Y, Z) + \eta(Y)\eta(Z)\}$$  \hspace{1cm} (28)

Differentiating this and using (3) we get

$$(\nabla_X \mathcal{L}_V g)(Y, Z) = -c\{\eta(Z)g(Y, \varphi X + \varphi hX) + \eta(Y)g(Z, \varphi X + \varphi hX)\}$$  \hspace{1cm} (29)
Equation (10) can be written

\[(\nabla_X \mathcal{L}_V g)(Y, Z) = \alpha((\mathcal{L}_V \nabla)(X, Y), Z) + \beta((\mathcal{L}_V \nabla)(X, Z), Y) \] (30)

A straightforward computation using (29) and (30) shows

\[(\mathcal{L}_V \nabla)(Y, Z) = -c\{\eta(Z)\varphi Y + \eta(Y)\varphi Z + g(Y, \varphi h Z)\xi\} \]

Its covariant differentiation and use of (2) provides

\[(\nabla_X \mathcal{L}_V \nabla)(Y, Z) = -c\{\eta(Z)(\nabla_X \varphi)Y + \eta(Y)(\nabla_X \varphi)Z
- g(Z, \varphi X + \varphi h X)\varphi Y - g(Y, \varphi X + \varphi h X)\varphi Z
- g(\varphi h Y, Z)(\varphi X + \varphi h X) + g((\nabla_X \varphi h)Y, Z)\xi\} \]

Using this in the commutation formula (13) for a Riemannian manifold, contracting at \(X\), and using equations (2), (3) and also the well known formula: 
\((\text{div}\varphi)X = -2n\eta(X)\) for a contact metric (see [1]), we find

\[(\mathcal{L}_V \text{Ric})(Y, Z) = c\{-2g(Y, Z) + 2g(h Y, Z)
+ 2g((\nabla_X \varphi h)Y, Z)\xi\} \]

Also, Lie-differentiating (9) along \(V\), and using Lemma 1 we have

\[(\mathcal{L}_V \text{Ric})(Y, Z) = (V\alpha + c\alpha)g(Y, Z) + (V\beta + c(\alpha + 2\beta))\eta(Y)\eta(Z) \] (32)

Comparing the previous two equations shows that

\[\begin{align*}
[V\alpha + c(\alpha + 2)]g(Y, Z) + [V\beta + c(\alpha + 2\beta - 2(2n + 1))]\eta(Y)\eta(Z)
& - c[2g(h Y, Z) - g((\nabla_X \varphi h)Y, Z)] = 0
\end{align*} \]

On one hand, we substitute \(Y = Z = \xi\) in the above equation getting one equation, and on the other hand, we contract the above equation (noting that both \(h\) and \(\varphi h\) are trace-free) getting another equation. Solving the two equations we obtain

\[V\alpha + c(\alpha + 2) = 0, \quad V\beta + c(\alpha + 2\beta - 4n - 2) = 0 \] (33)

The \(g\)-trace of equation (9 ) gives the scalar curvature

\[r = (2n + 1)\alpha + \beta \] (34)
The divergence of (9) along with the contracted second Bianchi identity yields 
\[ dr = 2d\alpha + 2(\xi\beta)\eta. \]
Taking its exterior derivative, and then exterior product with \( \eta \) we have \( (\xi\beta)\eta \wedge d\eta = 0 \). As \( \eta \wedge d\eta \) vanishes nowhere on \( M \), we find \( \xi\beta = 0 \) whence \( dr = 2d\alpha \). Hence \( V\alpha = Vr = 0 \), by hypothesis. Thus, it follows from (34) that \( V\beta = 0 \). Consequently, equations (33) reduce to: \( c(\alpha + 2) = 0 \) and \( c(\alpha + 2\beta - 4n - 2) = 0 \), and hence imply that, either \( c = 0 \) in which case \( V \) is an infinitesimal automorphism, or \( \alpha = -2 \) and \( \alpha + 2\beta = 4n + 2 \). In the second case, adding the two equations gives \( \alpha + \beta = 2n \).

But, from equation (9) we have \( \alpha + \beta = Tr.l \). Therefore, \( Tr.l = 2n \), and applying equation (6) we obtain \( h = 0 \), i.e. \( M \) is \( K \)-contact. As \( \alpha = -2 \), the \( \eta \)-Einstein structure is \( D \)-homothetically fixed, completing the proof.

4 An Explicit Example

An explicit example of non-trivial Ricci soliton as a Sasakian metric is the \((2n+1)\)-dimensional Heisenberg group \( \mathcal{H}^{2n+1} \) (which arose from quantum mechanics) of matrices of type
\[
\begin{bmatrix}
1 & Y & z \\
O^t & I & X^t \\
0 & O & 1
\end{bmatrix},
\]
where \( X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n), O = (0, \ldots, 0) \in \mathbb{R}^n, z \in \mathbb{R} \). As a manifold, this is just \( R^{2n+1} \) with coordinates \((x^i, y^i, z)\) where \( i = 1, \ldots, n \), and has the left-invariant Sasakian structure \((\eta, \xi, \varphi, g)\) defined by \( \eta = \frac{1}{2}(dz - \sum_{i=1}^n y^i dx^i), \xi = 2 \frac{\partial}{\partial z}, \varphi(\frac{\partial}{\partial x^i}) = -\frac{\partial}{\partial y^i}, \varphi(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}, \varphi(\frac{\partial}{\partial z}) = 0 \), and the Riemannian metric \( g = \eta \otimes \eta + \frac{1}{4} \sum_{i=1}^n ((dx^i)^2 + (dy^i)^2) \). Its \( \varphi \)-sectional curvature (i.e. the sectional curvature of plane sections orthogonal to \( \xi \)) is equal to \(-3\), so its Ricci tensor satisfies equation (20), as shown by Okumura [11], and hence \( \mathcal{H}^{2n+1} \) is a \( D \)-homothetically fixed null \( \eta \)-Einstein manifold. Setting \( V = \sum_{i=1}^n (V^i \frac{\partial}{\partial x^i} + \bar{V}^i \frac{\partial}{\partial y^i}) + V^z \frac{\partial}{\partial z}, \) using equations: \( \mathcal{L}_V \xi = 4(n+1)\xi, \mathcal{L}_V \varphi = 0 \) obtained in the proof of Theorem 1, and the aforementioned actions of \( \varphi \) on the coordinate basis vectors, shows that \( V^i \) and \( \bar{V}^i \) do not depend on \( z \) and yields the PDEs:

\[
\begin{align*}
\frac{\partial V^i}{\partial x^j} &= \frac{\partial V^i}{\partial y^j}, \quad \frac{\partial V^i}{\partial y^j} = -\frac{\partial V^i}{\partial x^j}, \quad y^j \frac{\partial V^i}{\partial y^j} = \frac{\partial V^z}{\partial y^j} \\
V^j &= y^j \frac{\partial V^z}{\partial z} - y^i \frac{\partial V^i}{\partial y^j}, \quad \frac{\partial V^z}{\partial z} = -4(n + 1)
\end{align*}
\]
The last equation readily integrates as \( V^z = -4(n + 1)z + F(x^i, y^i) \). For a special solution, assuming \( F = 0 \), \( V^i = cx^i \), \( V^i = cy^i \) and substituting in the above PDEs, we get \( c = -2(n + 1) \), and hence the Ricci soliton vector field \( V = -2(n + 1)(x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} + 2z \frac{\partial}{\partial z}) \). For dimension 3, this reduces to \( V = -4(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2z \frac{\partial}{\partial z}) \) which occurs on p. 37 of [4] without the factor 4, but gets adjusted with our \( \lambda = 6 \) which is 4 times their \( \lambda = 3/2 \).

**Remark 4** Another conclusion that we draw for Theorem 1 is the following: The value \(-2n\) for the scalar curvature \( r \) obtained during the proof, and the equation (17) show that the generalized Tanaka-Webster scalar curvature \([1]\) \( W = r - \text{Ric} (\xi, \xi) + 4n \) vanishes.

**Acknowledgment:** We thank the referee for valuable suggestions. R.S. was supported by University Research Scholar grant. This work is dedicated to Bhagawan Sri Sathya Sai Baba and Sri Ramakrishna Paramahansa.

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