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Amalendu Ghosh
Krishnagar Government College, India

Ramesh Sharma
University of New Haven, rsharma@newhaven.edu

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Contact Hypersurfaces Of A Bochner-Kaehler Manifold

Amalendu Ghosh[1] and Ramesh Sharma[2]

[1]*Department Of Mathematics, Krishnagar Government College, Krishnanagar 741101, West Bengal, India.*
E-mail: aghosh_70@yahoo.com

[2]*Department Of Mathematics, University Of New Haven, West Haven CT 06516, USA.*
E-mail: rsharma@newhaven.edu

Abstract: We have studied contact metric hypersurfaces of a Bochner-Kaehler manifold and obtained the following two results: (1) A contact metric constant mean curvature (CMC) hypersurface of a Bochner-Kaehler manifold is a (k, μ)-contact manifold, and (2) If M is a compact contact metric CMC hypersurface of a Bochner-Kaehler manifold with a conformal vector field V that is neither tangential nor normal anywhere, then it is totally umbilical and Sasakian, and under certain conditions on V, is isometric to a unit sphere.

Keywords: Bochner-Kaehler manifold, Contact metric hypersurface, Constant mean curvature, Conformal vector field.

MS Classification: 53B25, 53C55, 53 C15.

1 Introduction

Bochner curvature tensor was introduced in 1948 by S. Bochner [4] as a Kaehlerian analogue of the Weyl conformal tensor. It was shown by S.M. Webster [17] that the fourth order Chern-Moser curvature tensor of CR-manifolds coincides with the Bochner tensor. A Kaehler manifold with vanishing Bochner curvature tensor is known as Bochner-Kaehler manifold. Bochner-Kaehler surface is nothing but a self-dual Kaehler surface in Penrose's twistor theory. Some topological obstructions to

Bochner-Kaehler metrics were studied by Chen in [6]. Just as a real space-form is conformally flat, a complex space-form is Bochner flat, i.e. Bochner-Kaehler (the converse does not need to hold). The product of two complex space-forms of constant holomorphic sectional curvatures c and $-c$ is non-Einstein Bochner-Kaehler. Though Bochner-Kaehler manifolds have been studied by quite a few geometers, nevertheless have received considerably less attention, compared to Kaehler metrics with vanishing scalar curvature and Kaehler-Einstein metrics. For details we refer to Bryant [5]. It is well known that a hypersurface M of a Kaehler manifold $\bar{M}$ admits an almost contact metric structure induced from the Hermitian structure of $\bar{M}$. Okumura [13] studied and classified such hypersurfaces, mainly when the ambient space is a complex space-form. Generalizing the following result of Sharma [15] "The contact metric hypersurface of a complex space-form is a (k, μ)-contact manifold", we prove the following main result of this paper.

Theorem 1 *A contact metric constant mean curvature hypersurface of a Bochner-Kaehler manifold is a (κ, μ)-contact manifold.*

Finally, we consider the case when the ambient space admits a conformal vector field and provide the following extrinsic characterization of a Sasakian manifold.

Theorem 2 *Let M be a compact contact metric constant mean curvature hypersurface of a Bochner-Kaehler manifold $\bar{M}$ admitting a conformal vector field V which is neither tangential nor normal anywhere on M. Then M is Sasakian and totally umbilical in $\bar{M}$, and the component U of V, tangential to M is conformal on M. Further, (i) if U is non-Killing and dim.$M > 3$, then M is isometric to the unit sphere S^{2n+1}, and (ii) if V is closed, then for any dimension, M is isometric to S^{2n+1}.*

2 Contact Metric Hypersurfaces Of A Kaehler Manifold

A $(2n+1)$-dimensional smooth manifold M is said to be a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Given a contact 1-form η, there exists a unique vector field ξ such that $(d\eta)(\xi, X)= 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle

D $(\eta = 0)$, one obtains a Riemannian metric g and a (1,1)-tensor field φ such that

$$(d\eta)(X, Y) = g(X, \varphi Y), \eta(X) = g(X, \xi), \varphi^2 = -I + \eta \otimes \xi \qquad (1)$$

g is called an associated metric of η and (φ, η, ξ, g) a contact metric structure. The operators $h = \frac{1}{2}\pounds_\xi \varphi$ and $l = R(., \xi)\xi$ are self-adjoint and satisfy: $h\xi = 0$ and $h\varphi = -\varphi h$. Furthermore, h, $h\varphi$ are trace-free. Following formulas hold on a contact metric manifold.

$$\nabla_X \xi = -\varphi X - \varphi h X \qquad (2)$$

$$l - \varphi l \varphi = -2(h^2 + \varphi^2) \qquad (3)$$

If the associated CR-structure on M is integrable, then M is called a contact strongly pseudo-convex integrable CR manifold. This CR integrability condition was shown by Tanno [16] to be equivalent to

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \qquad (4)$$

and holds on a 3-dimensional contact metric manifold. A contact metric manifold (M, g) is said to be K-contact if ξ is Killing (equivalently, $h = 0$), and Sasakian if the almost Kaehler structure on the cone $M \times R$ with metric $dr^2 + r^2 g$ is Kaehler. Sasakian manifolds are K-contact and 3-dimensional K-contact manifolds are Sasakian. For details we refer to Blair [1]. In [2] Blair, Koufogiorgos and Papantoniou introduced a class of contact metric manifolds $M^{2n+1}(\eta, \xi, g, \varphi)$ satisfying the nullity condition:

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \qquad (5)$$

for real constants k and μ. Such manifolds are known as (k, μ)-contact manifolds, and satisfy: $k \leq 1$, equality holding when M is Sasakian.
Let M be an isometrically embedded orientable hypersurface of a Kaehler manifold $\bar{M}$ of real dimension $2n + 2$ and with complex structure tensor $J : J^2 = -I$ and the Hermitian metric g. The induced metric on M will also be denoted by g. If N denotes the unit normal vector field to M, we set

$$JN = \xi \qquad (6)$$

$$JX = \varphi X - \eta(X)N, \qquad (7)$$

where φ and η denote a $(1,1)$-tensor field and a 1-form respectively, and X an arbitrary vector field tangent to M. The Gauss and Weingarten formulas are

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \bar{\nabla}_X N = -AX$$

where X, Y denote arbitrary vector fields tangent to M, ∇ and $\bar{\nabla}$ the Riemannian connections of M and $\bar{M}$ respectively, and A the Weingarten operator. Differentiating (1) along an arbitrary vector field X tangent to M, using the Weingarten formula, and comparing tangential parts gives

$$\nabla_X \xi = -\varphi AX. \tag{8}$$

One can easily verify using (6) and (7) that (φ, ξ, η, g) defines the almost contact metric structure. We now assume that the almost contact metric structure induced on M is a contact metric structure. Using the formula (2) in (8) yields

$$A\xi = (Tr.A - 2n)\xi. \tag{9}$$

$$AX = X + hX + (Tr.A - 2n - 1)\eta(X)\xi. \tag{10}$$

which were derived in [15]. Next, differentiating (7) along M, and using (10) gives equation (4). Hence M is contact strongly pseudo-convex integrable CR manifold. We denote the Ricci tensor of M, of types $(0,2)$ and $(1,1)$ by S and Q respectively, and the scalar curvature by r of M. Corresponding objects of $\bar{M}$ are denoted by the same letters with overbars. Recall the Gauss equation

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) \\ +g(AX,Z)g(AY,W) &- g(AY,Z)g(AX,W). \end{aligned} \tag{11}$$

and contract it as

$$\begin{aligned} \bar{S}(Y,Z) - g(\bar{R}(N,Y)Z,N) &= S(Y,Z) \\ +g(AX,AZ) &- (Tr.A)g(AY,Z). \end{aligned} \tag{12}$$

For a Bochner-Kaehler manifold $\bar{M}$, the Bochner curvature tensor B (see [19]) vanishes, i.e. for arbitrarty vector fields $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ on $\bar{M}$, we have

$$\begin{aligned}
0 &= g(B(\bar{X},\bar{Y})\bar{Z},\bar{W}) = g(\bar{R}(\bar{X},\bar{Y})\bar{Z},\bar{W}) - \frac{1}{2n+6}[g(\bar{Y},\bar{Z})g(\bar{Q}\bar{X},\bar{W}) \\
&- g(\bar{Q}\bar{X},\bar{Z})g(\bar{Y},\bar{W}) + g(J\bar{Y},\bar{Z})g(\bar{Q}J\bar{X},\bar{W}) - g(\bar{Q}J\bar{X},\bar{Z})g(J\bar{Y},\bar{W}) \\
&+ g(\bar{Q}\bar{Y},\bar{Z})g(\bar{X},\bar{W}) - g(\bar{X},\bar{Z})g(\bar{Q}\bar{Y},\bar{W}) + g(\bar{Q}J\bar{Y},\bar{Z})g(J\bar{X},\bar{W}) \\
&- g(J\bar{X},\bar{Z})g(\bar{Q}J\bar{Y},\bar{W}) - 2g(J\bar{X},\bar{Q}\bar{Y})g(J\bar{Z},\bar{W}) \\
&- 2g(J\bar{X},\bar{Y})g(\bar{Q}J\bar{Z},\bar{W}) + \frac{\bar{r}}{(2n+4)(2n+6)}[g(\bar{Y},\bar{Z})g(\bar{X},\bar{W}) \\
&- g(\bar{X},\bar{Z})g(\bar{Y},\bar{W}) + g(J\bar{Y},\bar{Z})g(J\bar{X},\bar{W}) \\
&- g(J\bar{X},\bar{Z})g(J\bar{Y},\bar{W}) - 2g(J\bar{X},\bar{Y})g(J\bar{Z},\bar{W})] \qquad (13)
\end{aligned}$$

3 Proofs Of The Results

Lemma 1 *For a contact metric hypersurface of a Kaehler manifold,*

$$(a)\bar{S}(\varphi Y, Z) + \bar{S}(Y, \varphi Z) = \eta(Y)g(\bar{Q}N, Z) + \eta(Z)g(\bar{Q}N, Y)$$

$$(b)g(\xi, \bar{Q}N) = 0.$$

Proof: Since $\overline{M}$ is Kaehler, we have $\bar{S}(JY, Z) + \bar{S}(Y, JZ) = 0$. The use of (7) in this gives (a). Substituting ξ for Y and Z in (a) yields (b).

Lemma 2 *If f is a smooth function on a contact metric manifold M such that $df = (\xi f)\eta$ (d denoting exterior derivation), then f is constant on M.*

Proof: Taking the exterior derivative of the differential condition mentioned in the hypothesis gives $d(\xi f) \wedge \eta + (\xi f) d\eta = 0$. Taking its wedge product with η we find $(\xi f)(d\eta) \wedge \eta = 0$. As $(d\eta) \wedge \eta$ is nowhere vanishing on M (otherwise the definition of contact structure would break down), we conclude that $\xi f = 0$ on M. Consequently, $df = 0$ on M, and hence f is constant on M, completing the proof.

Lemma 3 *For a contact metric hypersurface M of a Bochner-Kaehler manifold, the following conditions are equivalent:*
(a) For any vector field X tangent to M, $g(\bar{Q}N, X) = 0$
(b) ξ is an eigenvector of the Ricci operator Q at each point of M
(c) The mean curvature of M is constant.

Proof : Using equations (12), (13), and part (b) of Lemma 1 gives

$$\begin{aligned}(2n+5)\bar{S}(Y,Z) &= [\bar{S}(N,N)-\frac{\bar{r}}{2n+4}]g(Y,Z)-\frac{3\bar{r}}{2n+4}\eta(Y)\eta(Z)\\ &+ 3g(\bar{Q}Z,\xi)\eta(Y)+3g(\bar{Q}Y,\xi)\eta(Z)+(2n+6)[S(Y,Z)\\ &+ g(AY,AZ)-(Tr.A)g(AY,Z)]\end{aligned}$$

Now we replace Y by φY in the above equation to get one equation, and replace Z by φZ to get another equation. Adding these two equations, and using part (a) of Lemma 1 we obtain

$$\begin{aligned}&(2n+5)\{\eta(Y)g(\bar{Q}N,Z)+\eta(Z)g(\bar{Q}N,Y)\}=3g(\bar{Q}\varphi Z,\xi)\eta(Y)\\ &+3g(\bar{Q}\varphi Y,\xi)\eta(Z)+(2n+6)[S(\varphi Y,Z)+S(Y,\varphi Z)+g(A\varphi Y,AZ)\\ &+g(AY,A\varphi Z)-(Tr.A)g(A\varphi Y,Z)-(Tr.A)g(AY,\varphi Z). \qquad (14)\end{aligned}$$

Substituting ξ for Z, using (9) and part (b) of Lemma 1 yields

$$(n+1)\varphi\bar{Q}\xi=(n+3)\varphi Q\xi. \qquad (15)$$

We also note

$$g(\bar{Q}N,X)=-g(J\bar{Q}\xi,X)=-g(\varphi\bar{Q}\xi,X), \qquad (16)$$

The equations (15) and (16) show that (a) is equivalent to (b). Contracting the Codazzi equation: $\bar{R}(X,Y)N=(\nabla_Y A)X-(\nabla_X A)Y$ at X provides $\bar{S}(Y,N)=Y(Tr.A)-(divA)Y$. Equation (10) transforms the preceding equation into

$$\bar{S}(Y,N)=Y(Tr.A)-(\xi Tr.A)\eta(Y)-(divh)Y \qquad (17)$$

Using equation (4) and the formula $(div.h)\xi=0$ (easy to verify) for a contact metric shows

$$(divh\varphi)\varphi Y=-(divh)Y \qquad (18)$$

Let us assume (b), i.e. $Q\xi=(Tr.l)\xi$. Applying the formula: $(divh\varphi)Y=S(Y,\xi)-2n\eta(Y)$ (see [3]), we have $(divh\varphi)\varphi Y=S(\varphi Y,\xi)=0$. Hence equation (18) shows that $divh=0$. As $Q\xi=(Tr.l)\xi$ is equivalent to (a) [proven earlier], appealing to equation (17) we obtain $d(Tr.A)=(\xi Tr.A)\eta$. Application of Lemma 2 shows that $Tr.A$ is constant on M, proving $(b)\Rightarrow(c)$. For the converse, assume (c), i.e. $Tr.A$ constant. Then, we go back to equations (17) and (18) and use the formula

$(divh\varphi)Y = S(Y,\xi) - 2n\eta(Y)$ once again, getting $\bar{S}(X,N) = S(\varphi X,\xi) = -g(\varphi Q\xi, X)$. Using this in (16) we find $\varphi\bar{Q}\xi = \varphi Q\xi$. Finally, using this in (15) we conclude that $\varphi Q\xi = 0$ which implies (b), and complete the proof.

Lemma 4 *For a contact metric hypersurface M of a Bochner-Kaehler manifold $\bar{M}$,*

$$(a)(Q\varphi - \varphi Q) - (\eta\circ Q\varphi)\otimes\xi + \eta\otimes\varphi Q\xi = 2(Tr.A - 2)h\varphi.$$

$$(b) l\varphi - \varphi l = 2(Tr.A - 2n)h\varphi.$$

Proof Replacing Y, Z by $\varphi Y, \varphi Z$ respectively, in (14) and then using (10) we get (a). Using the formula:

$$Q\varphi - \varphi Q = l\varphi - \varphi l + 4(n-1)h\varphi + (\eta\circ Q\varphi)\otimes\xi - \eta\otimes\varphi Q\xi = 0,$$

for a contact pseudo-convex integrable CR-manifold (see [10]), and using it in (a) we obtain (b).

Proof Of Theorem 1. First, we use equation (13) to obtain

$$\begin{aligned} g(\bar{R}(X,Y)\xi, W) = \frac{1}{2n+6}[\eta(Y)g(\bar{Q}X,W) - g(\bar{Q}X,\xi)g(Y,W) \\ -g(\bar{Q}X,N)g(JY,W) + g(\bar{Q}Y,\xi)g(X,W) - \eta(X)g(\bar{Q}Y,W) \\ +g(\bar{Q}Y,N)g(JY,W) - 2g(JX,Y)g(\bar{Q}\xi,W)] \\ -\frac{\bar{r}}{(2n+6)(2n+4)}[\eta(Y)g(X,W) - \eta(X)g(Y,W)] \end{aligned} \quad (19)$$

for arbitrary vector fields X, Y, W tangent to M. By Lemma 3, the constant mean curvature hypothesis is equivalent to $g(\bar{Q}N, X) = 0$, i.e. $\bar{Q}N = fN$ for some function f on M. We also have $\bar{Q}\xi = J\bar{Q}N = f\xi$. Hence equation (19) reduces to

$$\begin{aligned} (2n+6)g(\bar{R}(X,Y)\xi, W) &= \eta(Y)g(\bar{Q}X,W) - \eta(X)g(\bar{Q}Y,W) \\ &+ \sigma[\eta(Y)g(X,W) - \eta(X)g(Y,W)] \end{aligned} \quad (20)$$

where $\sigma = f - \frac{\bar{r}}{2n+4}$. This shows that $g(\overline{R}(X,Y)\xi, W) = 0$, for any vector fields X, Y tangent to M and orthogonal to ξ. Next, substituting ξ for Z in (11), and using equations (9) and (20) we obtain $R(X,Y)\xi = 0$,

for any vector fields X, Y tangent to M and orthogonal to ξ. Hence by the result of Koufogiorgos-Stamatiou ([11]) we conclude that M is a (κ, μ)-space provided the dimension of M is ≥ 5.
It remains to consider the 3-dimensional case for which we know that

$$\begin{aligned} R(X,Y)Z &= g(QY,Z)X - g(QX,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{21}$$

Making use of the formula (3) and the formula $h^2 = (k-1)\varphi^2$ (for any 3-dimensional contact metric manifold [9], where k is a function $= \frac{Tr.l}{2}$) in part (b) of Lemma 4, we obtain

$$lY = -k\varphi^2 Y + (Tr.A - 2)hY \tag{22}$$

Differentiating this along an arbitrary vector field X, using (2) and then contracting the resulting equation at X with respect to a local orthonormal frame e_i, we find

$$\begin{aligned} g((\nabla_Y Q)\xi,\xi) - g((\nabla_\xi Q)Y,\xi) - g(R(Y,\varphi e_i + \varphi h e_i)\xi, e_i) \\ -g(R(Y,\xi)(\varphi e_i + \varphi h e_i), e_i) = -(\varphi^2 Y)\kappa + (Tr.A - 2)(divh)Y \end{aligned} \tag{23}$$

Using $Q\xi = (Tr.l)\xi$ and (2) we have $g((\nabla_Y Q)\xi,\xi) = 2(Yk)$, $g((\nabla_\xi Q)Y,\xi) = 2(\xi k)\eta(Y)$. We also had found during the proof of Lemma 3 that $div.h = 0$. Moreover, using (21) and $Q\xi = (Tr.l)\xi$ we compute

$$g(R(Y,\varphi e_i + \varphi h e_i)\xi, e_i) = -\eta(Y)Tr.(Q\varphi h)$$

and

$$g(R(Y,\xi)(\varphi e_i + \varphi h e_i), e_i) = 0$$

Utilizing all these findings in (23), we obtain

$$Yk - (\xi k)\eta(Y) + \eta(Y)Tr.(Q\varphi h) = 0$$

Taking $Y = \xi$ it is easy to see that $Tr.Q\varphi h = 0$. Hence $Yk = (\xi k)\eta(Y)$, i.e. $dk = (\xi k)\eta$. Applying Lemma 2, we conclude that k is constant. Thus, the hypothesis : $Q\xi = (Tr.l)\xi$ and (21) imply $R(X,Y)\xi = 0$, for any vector field X, Y orthogonal to ξ. Replacing X by $X - \eta(X)\xi$ and Y by $Y - \eta(Y)\xi$ (as these vector fields are orthogonal to ξ) we obtain

$$R(X,Y)\xi = \eta(Y)lX - \eta(X)lY$$

The use of (22) in the foregoing equation shows that M^3 is a (κ, μ) space with $\mu = (Tr.A - 2)$. This completes the proof.

Proof Of Theorem 2. As V is conformal on $\bar{M}$,

$$\pounds_V g = 2\rho g \tag{24}$$

We decompose the conformal vector field V along M orthogonally as

$$V = U + \alpha N \tag{25}$$

where U is the tangential part of V and α a smooth function on M. In view of the Lemma 3, the constant mean curvature hypothesis is equivalent to $\bar{S}(X, N) = 0$ for arbitrary vector field X tangent to M. Following the procedure given on pages 101-104 of Yano [18], we have

$$2n \int_M \bar{S}(U, N) dM = \int_M \alpha \sum_{i \neq j}^{2n+1} (k_i - k_j)^2 dM \tag{26}$$

where dM is the volume element of M, and k_i are the principal curvatures of M. As the left hand side of the above equation is zero, and α is nowhere zero on M (otherwise V would become tangent to M somewhere, contradicting our hypothesis), we conclude that $k_i = k_j$, i.e. M is totally umbilical. Hence, using (10) provides $A = I$, and $h = 0$, i.e. M is Sasakian. The conformal Killing equation (24), together with the Gauss and Weingarten formulas show that $\pounds_U g = 2(\rho + \alpha)g$, i.e. U is conformal on M. If U is homothetic, then U reduces to Killing, since M is compact. Hence, if U is not Killing, and $dim > 3$, then by the following theorem of Okumura [14] "A complete Sasakian manifold of dimension > 3 and admitting a non-Killing conformal vector field is isometric to a unit sphere", M is isometric to S^{2n+1} which proves (i). For (ii), we know from the following result of Goldberg [8]"A closed conformal vector field on a non-flat Kaehler manifold is homothetic and holomorphic" that V is homothetic. Hence $\nabla_X V = \rho X$ (ρ constant). Using the decomposition (25), and bearing in mind that X is arbitrary tangent vector on M, we immediately obtain $U = -D\alpha$ (D is the gradient operator of (M, g)) and $\nabla_X U = (\alpha + \rho)X$. We note that α cannot be constant on M, otherwise U would vanish on M turning V normal to M and thus contradicting our hypothesis. Thus obtain

$$\nabla_X D(\alpha + \rho) = -(\alpha + \rho)X \tag{27}$$

Hence, by Obata's theorem [12]"A complete Riemannian manifold of dimension > 2 is isometric to a sphere of radius $\frac{1}{c}$ if and only if it admits a non-trivial solution f of the differential equation $\nabla\nabla f = -c^2 fg$", we conclude that M is isometric to S^{2n+1}, completing the proof.

Remark. As indicated in [7], the Kaehler cone manifold $(M \times R^+, d(r^2\eta)$ with metric $dr \otimes dr + r^2 g$ over a Sasakian manifold (M, η, g) admits a conformal vector field $ar\partial_r - b\xi$ (for a, b real constants) which is nowhere tangent and nowhere normal to M and therefore serves as an example of the conformal vector field satisfying the hypothesis of Theorem 2.

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