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Sasakian manifolds with purely transversal Bach tensor

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We show that a $(2n+1)$-dimensional Sasakian manifold (M, g) with a purely transversal Bach tensor has constant scalar curvature $\geq 2n(2n+1)$, equality holding if and only if (M, g) is Einstein. For dimension 3, M is locally isometric to the unit sphere S^3. For dimension 5, if in addition (M, g) is complete, then it has positive Ricci curvature and is compact with finite fundamental group $\pi_1(M)$. *Published by AIP Publishing.*
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I. INTRODUCTION

In 1921, Bach[1] introduced a tensor to study the conformal relativity in the context of conformally Einstein spaces. This tensor is known as the Bach tensor and is a symmetric (0, 2)-tensor B on a pseudo-Riemannian manifold (M, g), defined by

$$B(X,Y) = \frac{1}{d-3}\sum_{i,j=1}^{d}((\nabla_{e_i}\nabla_{e_j}W)(X,e_i,e_j,Y) + \frac{1}{d-2}\sum_{i,j=1}^{d} Ric(e_i,e_j)W(X,e_i,e_j,Y), \tag{1.1}$$

where (e_i), $i = 1, \dots, d$, is a local orthonormal frame on (M, g), Ric is the Ricci tensor of type (0, 2), and W denotes the Weyl tensor of type (0, 4) defined by

$$W = R - \frac{2}{d-2}Ric \odot g + \frac{r}{(d-1)(d-2)} g \odot g \tag{1.2}$$

where $\odot$ is the Kulkarni-Nomizu product defined for two symmetric (0, 2)-tensors s and t as

$$(s \odot t)(X,Y,Z,W) = \frac{1}{2}\left[t(X,W)s(Y,Z) + t(Y,Z)s(X,W) - t(X,Z)s(Y,W) - t(Y,W)s(X,Z)\right],$$

where X, Y, Z, W denote arbitrary vector fields on M. This convention will be followed throughout this paper. We recall the Cotton tensor C which is a (0, 3)-tensor defined by

$$C(X,Y,Z) = (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) - \frac{1}{2(d-1)}[(Xr)g(Y,Z) - (Yr)g(X,Z)]. \tag{1.3}$$

In view of Eqs. (1.1) and (1.2), the Bach tensor can be expressed as (Chen and He[6])

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$$B(X,Y) = \frac{1}{d-2}\,[\sum_{i=1}^{d}((\nabla_{e_i}C)(e_i,X,Y) + \sum_{i,j=1}^{d} Ric(e_i,e_j)W(X,e_i,e_j,Y)\,]\,. \tag{1.4}$$

In dimension 3, the Weyl tensor W vanishes, and hence the Bach tensor expression reduces to

$$B(X,Y) = \sum_{i=1}^{3}((\nabla_{e_i}C)(e_i,X,Y). \tag{1.5}$$

The metric g is said to be Bach flat when $B = 0$. Einstein and locally conformally flat metrics are obviously Bach flat. For a 4-dimensional compact manifold, it is interesting to note that Bach flat metrics are precisely the critical points of the Weyl functional $\mathcal{W}(g) = \int_M |W_g|^2 dvol_g$.

An odd dimensional analog of the Kaehler geometry is the Sasakian geometry. The Kaehler cone over a Sasakian Einstein manifold is a Calabi-Yau manifold which has application in physics in superstring theory based on a 10-dimensional manifold that is the product of the 4-dimensional space-time and a 6-dimensional Ricci-flat Kaehler (Calabi-Yau) manifold (see the work of Candelas *et al.*[5]). The Sasakian geometry has been extensively studied since its recently perceived relevance in string theory. Sasakian Einstein metrics have received a lot of attention in physics, for example, p-brane solutions in superstring theory and Maldacena conjecture (AdS/CFS duality).[9] For details, see the studies of Boyer and Galicki,[3] Boyer, Galicki, and Matzeu,[4] and Cvetic *et al.*[7]

In this paper, we consider a Sasakian manifold (M, g) with a weaker condition on the Bach tensor, i.e., B is purely transversal, i.e., B has components only along the contact (transversal) subbundle D ($\eta = 0$). We note that this condition is equivalent to $B(\xi,.) = 0$ and obtain the following results.

Theorem 1.1. *Let (M, g) be a $(2n + 1)$-dimensional Sasakian manifold with a purely transversal Bach tensor. Then (i) g has constant scalar curvature $\geq 2n(2n+1)$, with equality holding if and only if g is Einstein, and (ii) the Ricci tensor of g has a constant norm.*

Proposition 1.1. Under the same hypothesis as in Theorem 1.1, for dimension 3, (M, g) is locally isometric to the unit sphere S^3, and for dimension 5, if in addition (M, g) is complete, then its Ricci curvature has positive constant eigenvalues. In the last case, (M, g) is compact with finite fundamental group.

II. A BRIEF REVIEW OF SASAKIAN GEOMETRY

A $(2n + 1)$-dimensional smooth manifold is said to be contact if it has a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ on M. For a contact 1-form η, there exists a unique vector field ξ such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, we obtain a Riemannian metric g and a (1,1)-tensor field φ such that

$$d\eta(X,Y) = g(X,\varphi Y), \eta(X) = g(X,\xi), \varphi^2 = -I + \eta\otimes\xi, \tag{2.1}$$

where g is called an associated metric of η and (φ,η,ξ,g) is called a contact metric structure. The contact metric structure on M is said to be Sasakian if the almost Kaehler structure on the cone manifold $(M \times R^+, r^2 g + dr^2)$ over M is Kaehler. For a Sasakian manifold,

$$\nabla_X\xi = -\varphi X, \tag{2.2}$$

$$Q\xi = 2n\xi, \tag{2.3}$$

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.4}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.5}$$

where ∇, R, and Q denote the Levi-Civita connection, curvature tensor, and (1,1)-Ricci tensor of g. For details, see Ref. 2. A Sasakian manifold M is said to be η-Einstein if the Ricci tensor can be written as

$$Ric(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y) \tag{2.6}$$

for some smooth functions α and β on M. It is well known (Yano and Kon[11]) that α and β are constant in dimension greater than 3 and are, respectively, equal to $\frac{r}{2n} - 1$ and $2n + 1 - \frac{r}{2n}$. Motivated by this result, Hasegawa and Nakane[8] studied the η-Einstein tensor S defined by

$$S = Ric - \alpha g - \beta\eta \otimes \eta. \tag{2.7}$$

Thus a Sasakian manifold is η-Einstein if and only if $S = 0$.

III. PROOFS OF THE THEOREMS

First, we prove the following lemma.

Lemma 3.1. Let $\{e_i : i = 1, \ldots, 2n+1\}$ *be a local orthonormal frame on the Sasakian manifold M. Then*

$$(i) \sum_{i=1}^{2n+1} g((\nabla_X Q)\varphi e_i, e_i) = 0, \tag{3.1}$$

$$(ii) \sum_{i=1}^{2n+1} g((\nabla_{e_i} Q)X, \varphi e_i) = \{r - 2n(2n+1)\}\eta(X) - \frac{1}{2}(\varphi X r). \tag{3.2}$$

Proof. We know that the Ricci operator Q commutes with φ, i.e., $Q\varphi = \varphi Q$ on a Sasakian manifold. Differentiating this along an arbitrary vector field X and using (2.4) provide

$$\begin{aligned} g(\nabla_X Q)\varphi Y, Z) + g(\nabla_X Q)Y, \varphi Z) = {} & g(X, QY)\eta(Z) - 2ng(X,Y)\eta(Z) \\ & + g(QX, Z)\eta(Y) - 2ng(X,Z)\eta(Y). \end{aligned}$$

Substituting e_i for Y and Z in the above equation, using (2.3), and summing over i give part (i). Next, substituting e_i for X and Z in the above equation and summing over i, we get

$$\begin{aligned} \sum_{i=1}^{2n+1}[g(\nabla_{e_i} Q)\varphi Y, e_i) + g(\nabla_{e_i} Q)Y, \varphi e_i)] = {} & \sum_{i=1}^{2n+1}[g(e_i, QY)\eta(e_i) - 2ng(e_i, Y)\eta(e_i) \\ & + g(Qe_i, e_i)\eta(Y) - 2ng(e_i, e_i)\eta(Y)]\,. \end{aligned}$$

Now the combined use of $\sum_{i=1}^{2n+1} g(\nabla_{e_i} Q)\varphi Y, e_i) = \frac{1}{2}(\varphi Y)r$ (which follows from the twice contracted second Bianchi identity: $divQ = \frac{1}{2}dr$) and (2.3) proves part (ii) of the lemma.

Proof of Theorem 1. As the dimension $d = 2n+1$, Eq. (1.4) becomes

$$\begin{aligned} B(X,Y) = \frac{1}{2n-1}\Big[& \sum_i (\nabla_{e_i} C)(e_i, X)Y \\ & + \sum_{i,j} g(Qe_i, e_j) g(W(X, e_i)e_j, Y)\Big] \end{aligned} \tag{3.3}$$

and in view of (1.2), the Weyl tensor takes the form

$$\begin{aligned} W(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}\{g(QY,Z)X - g(QX,Z)Y \\ + g(Y,Z)QX - g(X,Z)QY\} + \frac{r}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.4}$$

Substituting ξ for Z in the expression (1.3) for the Cotton tensor, we have

$$\begin{aligned} C(X,Y)\xi = {} & g((\nabla_X Q)Y, \xi) - g((\nabla_Y Q)X, \xi) \\ & - \frac{1}{4n}\{\eta(Y)(Xr) - \eta(X)(Yr)\}. \end{aligned} \tag{3.5}$$

Differentiating (2.3) along an arbitrary vector field X and using (2.2) shows

$$(\nabla_X Q)\xi = Q\varphi X - 2n\varphi X.$$

Using this and the Sasakian property $Q\varphi = \varphi Q$ in Eq. (3.5) provides

$$C(X,Y)\xi = 2g(Q\varphi X,Y) - 4ng(\varphi X,Y) - \frac{1}{4n}\{\eta(Y)(Xr) - \eta(X)(Yr)\}.$$

Differentiating the above equation along an arbitrary vector field Z and using (2.3), we find

$$\begin{aligned}(\nabla_Z C)(X,Y)\xi - C(X,Y)\varphi Z = {} & 2g((\nabla_Z Q)\varphi X,Y) + 2g(Q(\nabla_Z\varphi)X,Y)\\ & - 4ng((\nabla_Z\varphi)X,Y) - \frac{1}{4n}\{g(\nabla_Z Dr,X)\eta(Y) - (Xr)g(\varphi Z,Y)\\ & + (Yr)g(X,\varphi Z) - g(\nabla_Z Dr,Y)\eta(X)\},\end{aligned}$$

where Dr denotes the gradient of r. Combined use of (1.3), (2.3), and (2.4) transforms the above equation into

$$\begin{aligned}&(\nabla_Z C)(X,Y)\xi - g((\nabla_X Q)Y,\varphi Z) + g((\nabla_Y Q)X,\varphi Z)\\ &+\frac{1}{4n}\{g(Y,\varphi Z)(Xr) - g(X,\varphi Z)(Yr)\} = 2g((\nabla_Z Q)\varphi X,Y)\\ &-2\eta(X)g(QZ,Y) + 4ng(Y,Z)\eta(X) - \frac{1}{4n}\{g(\nabla_Z Dr,X)\eta(Y)\\ &- (Xr)g(\varphi Z,Y) + (Yr)g(X,\varphi Z) - g(\nabla_Z Dr,Y)\eta(X)\}.\end{aligned} \tag{3.6}$$

Substituting e_i for X and Z in the preceding equation and using Lemma 3.1, we get the relation

$$\begin{aligned}\sum_{i=1}^{2n+1}(\nabla_{e_i}C)(e_i,Y)\xi = {} & 3\{r - 2n(2n+1)\}\eta(Y) - \frac{3}{2}g(\varphi Y,Dr)\\ & -\frac{1}{4n}\{(divDr)\eta(Y) - g(\nabla_\xi Dr,Y)\}.\end{aligned} \tag{3.7}$$

Next, substituting ξ for Z in (3.4), using the formulas (2.3) and (2.5) and subsequently, operating by the Ricci operator Q, we find that

$$\begin{aligned}QW(X,Y)\xi = {} & \frac{1}{2n-1}\{\eta(X)Q^2Y - \eta(Y)Q^2X\}\\ & + \frac{r-2n}{2n(2n-1)}\{\eta(Y)QX - \eta(X)QY\}.\end{aligned} \tag{3.8}$$

Substituting e_i for Y in the above equation, taking inner product with e_i, summing over i, and using (2.3), we obtain

$$\sum_{i=1}^{2n+1} g(QW(X,e_i)\xi,e_i) = \frac{|Q|^2 - 4n^2}{2n-1}\eta(X) - \frac{(r-2n)^2}{2n(2n-1)}\eta(X). \tag{3.9}$$

Now we notice that the last term of the Bach tensor in (3.3) can be written as

$$g(Qe_i,e_j)g(W(X,e_i)e_j,Y) = -g(W(X,e_i)Y,Qe_i) = -g(QW(X,e_i)Y,e_i)$$

and hence (3.3) assumes the form

$$B(X,Y) = \frac{1}{2n-1}[\sum_i(\nabla_{e_i}C)(e_i,X,Y) - \sum_i g(QW(X,e_i)Y,e_i)]. \tag{3.10}$$

Here we substitute ξ for Y in the preceding equation, use the hypothesis $B(X,\xi) = 0$, along with Eqs. (3.7) and (3.9) so as to get

$$\begin{aligned}&3\{r - 2n(2n+1)\}\eta(X) - \frac{3}{2}g(\varphi X,Dr) - \frac{1}{4n}\{(divDr)\eta(X)\\ &- g(\nabla_\xi Dr,X)\Big\} - \frac{|Q|^2 - 4n^2}{2n-1}\eta(X) + \frac{(r-2n)^2}{2n(2n-1)}\eta(X) = 0.\end{aligned} \tag{3.11}$$

Replacing the arbitrary X by φX in the above equation entails

$$\nabla_\xi Dr = -6n\varphi Dr. \tag{3.12}$$

As ξ is Killing, we have $\pounds_\xi r = 0$. Operating exterior derivative d on it and noting that d commutes with $\pounds_\xi$, we get $\pounds_\xi dr = 0$ which, in turn, implies $\pounds_\xi Dr = 0$. Use of (2.2) in the preceding equation shows $\nabla_\xi Dr = -\varphi Dr$. Combining this with (3.12) yields $\varphi Dr = 0$. Operating this by φ and noting $\xi r = 0$, we conclude that the scalar curvature is constant. Thus, using (3.11), we compute

$$|Ric|^2 = |Q|^2 = 4n^2 + 3(2n-1)(r - 2n(2n+1)) + \frac{1}{2n}(r-2n)^2. \tag{3.13}$$

Hence the Ricci operator has a constant norm, proving part (ii). Through (3.13), we obtain the squared norm of the Einstein deviation tensor as follows:

$$|Ric - \frac{r}{2n+1}g|^2 = [r - 2n(2n+1)][\frac{r-2n(2n+1)}{2n(2n+1)} + 14n - 3]. \tag{3.14}$$

A straightforward computation of the squared norm of the η-Einstein tensor S (described at the end of Sec. II) using (3.13) provides

$$|S|^2 = 3(2n-1)[r - 2n(2n+1)]. \tag{3.15}$$

This shows that $r \geq 2n(2n+1)$. The equality case $r = 2n(2n+1)$ implies, by virtue of (3.14), that $Ric = \frac{r}{2n+1}g$, i.e., g is Einstein. The converse is obvious. This completes the proof of Theorem 1.

Proof of Proposition 1. All the equations in the proof of Theorem 1 are applicable. For the 3 dimensional case, Eq. (1.5) and the hypothesis $B(X, \xi) = 0$ imply $\sum_{i=1}^{3}((\nabla_{e_i}C)(e_i, X, \xi) = 0$. Using this in Eq. (3.7) and noting that $n = 1$ and r is constant immediately provide $r = 6$. Appealing now to Eq. (3.14), we get $Ric = \frac{r}{3}g$, i.e., g is Einstein. Hence, as M is 3-dimensional, we conclude that it has constant curvature 1 and hence locally isometric to the unit sphere S^3.

Now we turn our attention to dimension 5, for which $n = 2$. As the Ricci operator Q is self-adjoint, it is diagonalizable and hence we can have a local orthonormal φ-frame $e_1, e_2, \varphi e_1, \varphi e_2, \xi$ such that

$$Qe_1 = r_1 e_1, Qe_2 = r_2 e_2, Q\varphi e_1 = r_1 \varphi e_1, Q\varphi e_2 = r_2 \varphi e_2. \tag{3.16}$$

We already know from (2.3) that $Q\xi = 4\xi$. Thus, $r = 2(r_1 + r_2) + 4$ and $|Q|^2 = 2(r_1^2 + r_2^2) + 16$. As already shown, r is constant and $\geq 2n(2n+1)$, ≥ 20 because $n = 2$. So, it turns out that

$$r_1 + r_2 = c \geq 8, \tag{3.17}$$

for a positive constant c. Further, from Eq. (3.13), we find that

$$4r_1 r_2 = (r_1 + r_2 - 9)^2 + 63. \tag{3.18}$$

Hence $r_1 r_2 \geq \frac{63}{4}$. This, in conjunction with (3.18), implies that both r_1 and r_2 are positive. Furthermore, combining Eqs. (3.17) and (3.18), r_1 and r_2 are positive constants with values $\frac{1}{2}(c + 3\sqrt{2(c-8)})$ and $\frac{1}{2}(c - 3\sqrt{2(c-8)})$. By hypothesis, (M, g) is complete and hence by Myers' theorem, we conclude that it is compact and has finite fundamental group. This completes the proof.

Remark 1. For the 5-dimensional case of Proposition 1.1, we also conclude from the well-known Bochner's theorem "If M is a compact Riemannian manifold that has positive Ricci curvature, then the first Betti number $b_1(M) = 0$" (see Ref. 10) that $b_1(M) = 0$.

Remark 2. In Ref. 12, Zhang proved the following result: "If a compact Sasakian manifold with constant scalar curvature has quasi-positive holomorphic bisectional transverse curvature, then it is η-Einstein." Recall that the holomorphic bisectional transverse curvature is defined as $g(R^T(X, JX)JY, Y)$, where X and Y are unit vectors in two φ-invariant planes in the contact sub-bundle D defined by $\eta = 0$ and R^T is the curvature tensor of a transverse Levi-Civita connection of the transverse metric g^T (the restriction of g to D). This curvature is quasi-positive if it is non-negative everywhere and strictly positive somewhere. If we impose this condition on the 5-dimensional case of Proposition 1.1, then (M, g) becomes Einstein because in our case the η-Einstein implies Einstein.

Remark 3. We recall the following result of Hasegawa and Nakane:[8] "A 5-dimensional Sasakian manifold with constant scalar curvature $\neq -4$ and vanishing contact Bochner curvature tensor is a space of constant φ-sectional curvature." Applying this to the 5-dimensional case of Propositions 1.1 and noting that r is constant ≥ 20, we conclude that a complete 5-dimensional Sasakian manifold with a purely transversal Bach tensor and vanishing contact Bochner curvature tensor is locally isometric to the unit sphere S^5.

Remark 4. In the 3-dimensional case, (M, g) becomes locally isometric to a unit 3-sphere and hence the Bach tensor vanishes completely. However, we note that this does not happen in higher dimensions. So it would be desirable to examine the impact of the full Bach flat condition for dim . > 3, in which case we anticipate that the Sasakian metric would become Einstein.

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