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Cosmological Models Through Spatial Ricci Flow

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Abstract. We consider the synchronization of the Einstein's flow with the Ricci-flow of the standard spatial slices of the Robertson-Walker space-time and show that associated perfect fluid solution has a quadratic equation of state and is either spherical and collapsing, or hyperbolic and expanding.

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1 Introduction

Let us consider the Robertson-Walker (R-W) space-time $(M, g_{\alpha\beta})$ with the line-element:

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + g_{ij}dx^i dx^j \tag{1}$$

with the spatial time-dependent metric

$$g_{ij} = f^2(t)\gamma_{ij} \tag{2}$$

where $f(t)$ is the warping (scale) function, and γ_{ij} is the 3-dimensional fixed (time-independent) Riemannian metric of constant curvature k. The space-time indices α, β run over 0,1,2,3 such that $x^0 = t$, and the spatial indices i, j run over 1,2,3. The exact solution of Einstein's equations for a perfect fluid with R-W metric is generally found by assuming an equation of state (generally, linear). The standard solution is the Friedman's solution for dust. The purpose of this note is to examine the effect of synchronizing the Einstein flow of the R-W space-time along the evolution vector field $\frac{\partial}{\partial t}$, with the Ricci-flow (see Chow et al. [1] and Kouneiher [2]) of the time-dependent metric g_{ij}, i.e.

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \quad (3)$$

(first introduced by Hamilton, and later used by Perelman to prove Poincare conjecture), and to show that the associated perfect fluid has a quadratic equation of state, and evolves over a finite time domain for $k > 0$ and $k < 0$. The $k = 0$ corresponds to the static degenerate solution $f(t) = f_0$.

2 Basic Kinematic And Geometric Equations

As the metric g_{ij} is homothetic to γ_{ij}, we first note that their Ricci tensors are equal, i.e. $R_{ij} = R^{\gamma}_{ij}$. As γ has constant curvature k, we have

$$R^{\gamma}_{ij} = 2k\gamma_{ij} \quad (4)$$

Thus (3) provides

$$\frac{d}{dt} f^2(t) = -4k$$

whence

$$f^2(t) = c - 4kt \quad (5)$$

for a constant c. Obviously, $c = f_0^2$ where $f_0 = f(0)$. Thus equation (5) becomes

$$f^2(t) = -4k(t - \frac{f_0^2}{4k}) \quad (6)$$

We infer from equation (6) that $k = 0$ corresponds to the degenerate solution $f(t) = f_0$, and so we will be discussing the two cases: I. $k > 0$, and II. $k < 0$.

In case I, the graph of $f(t)$ versus t is the upper half of a parabola with vertex $\frac{f_0^2}{4k}$ and opening to the left, and therefore the solution $f(t)$ is singular at the vertex $t = \frac{f_0^2}{4k}$, [spherical collapsing solution] whereas in case II, the solution is singular at the vertex $t = \frac{f_0^2}{4k}$ [expanding hyperbolic solution]. In the next section we will study the constraints on the domains of the Ricci flow, imposed by the energy conditions.

3 Perfect Fluid Solution

First we notice from equation (6) that $\frac{\dot{f}}{f} = -2\frac{k}{f^2}$ and $\frac{\ddot{f}}{f} = -\frac{4k^2}{f^4}$. The standard computations in warped product geometry (see O'Neill [4] for details) and the use of equation (4) lead us to the following components of the space-time Ricci tensor :

$$\bar{R}_{ij} = (\frac{2k}{f^2} + \frac{4k^2}{f^4})g_{ij} \tag{7}$$

$$\bar{R}_{i0} = 0 \tag{8}$$

$$\bar{R}_{00} = \frac{12k^2}{f^4} \tag{9}$$

At this point, we would like to examine the perfect fluid solution corresponding to the Robertson-Walker metric (1) with (2) and (6) through the spatial Ricci flow. The Einstein's equations for a perfect fluid solution are:

$$\bar{R}_{\alpha\beta} = 4\pi(\rho - p)g_{\alpha\beta} + 8\pi(\rho + p)u_\alpha u_\beta \tag{10}$$

where u_α the covariant components of the 4-velocity, ρ the energy density, and p the pressure of the fluid. Comparing the components of $\bar{R}_{\alpha\beta}$ from (10), with equations (7, 8, 9) provides the following solutions for ρ and p:

$$\frac{8\pi\rho}{3} = \frac{4k^2 + kf^2}{f^4}, \quad 8\pi p = \frac{4k^2 - kf^2}{f^4} \tag{11}$$

Obviously, the vacuum solution ($\rho = 0, p = 0$) is ruled out, because then $k = 0$ and $f = f_0$, as expected from the fact that Ricci flow is parabolic and vacuum Einstein's equations are hyperbolic.

We observe by eliminating f from the equations (11) that

$$(\rho - 3p)^2 = \frac{3}{16\pi}(\rho + 3p) \tag{12}$$

i.e. ρ and p are quadratically related. This provides a new perfect fluid solution for which the relationship between p and ρ is quadratic.

We can easily check that the dominant energy condition (DEC): $\rho > 0$, $\rho > p$ (see Stephani et al. [5]) holds provided

$$2k^2 + kf^2 > 0 \tag{13}$$

We observe that, for case $k>0$, the DEC (13) is obviously satisfied. For case $k<0$, the DEC requires

$$2k+f^2<0. \tag{14}$$

We also find, by a straight computation using (11) that $\frac{dp}{d\rho}=\frac{8k-f^2}{3(8k+f^2)}$. For a physically realistic solution, $0\leq\frac{dp}{d\rho}=c_s^2<1$, where c_s is the speed of sound and velocity of light is 1 in relativistic units (for example, see Mitra [3]). Taking into account this condition, the DEC, and that $t<\frac{f_0^2}{4k}$ for $k>0$ and $t>\frac{f_0^2}{4k}$ for $k<0$ we find the two solutions with following domains of development:

Case I (Spherical collapsing model):

$$k>0: \quad -2+\frac{f_0^2}{4k}<t<\frac{f_0^2}{4k} \tag{15}$$

Case II (Hyperbolic expanding model):

$$k<0: \quad \frac{f_0^2}{4k}<t<\frac{f_0^2}{4k}+\frac{1}{2} \tag{16}$$

In particular, choosing $k=\frac{1}{4}, f_0=1$ in case I gives the domain $-1<t<1$. In case II, Choosing $k=-\frac{1}{4}, f_0=\frac{1}{2}$ gives the domain $-\frac{1}{4}<t<\frac{1}{4}$.

Concluding Remark: Exact solutions for a perfect fluid Robertson-Walker space-time are usually acquired by assuming a barotropic equation of state, such as the linear equation $p=(\gamma-1)p$, where $1\leq\gamma\leq2$. The standard Friedmann model corresponds to $\gamma=1$, i.e. $p=0$ (dust). In this paper we have used the Ricci flow of spatial metric of the warped product space-time, obtaining a perfect fluid solution which evolves over a finite time domain in both cases $k>0$ and $k<0$, and whose equation of state is non-linear (quadratic).

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