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Ramesh Sharma  
*University of New Haven, rsharma@newhaven.edu*

S Balasubramanian  
*Sri Sathya Sai Institute of Higher Learning*

N. Uday Kiran  
*Sri Sathya Sai Institute of Higher Learning*

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Some Remarks On Ricci Solitons

Ramesh Sharma\textsuperscript{a}, S. Balasubramanian\textsuperscript{b}, and N. Uday Kiran\textsuperscript{c}

\textsuperscript{a} Department of Mathematics, University of New Haven, West Haven, CT 06516, U.S.A. E-mail: rsharma@newhaven.edu

\textsuperscript{b} Department of Mathematics and Computer Science, Sri Sathya Sai Institute Of Higher Learning, Prasanthinilayam-515134, India, E-mail: sbalasubramanian@sssihl.edu.in

\textsuperscript{c} Department of Mathematics and Computer Science, Sri Sathya Sai Institute Of Higher Learning, Prasanthinilayam-515134, India, E-mail: nudaykiran@sssihl.edu.in

Abstract: We obtain an intrinsic formula of a Ricci soliton vector field and a differential condition for the non-steady case to be gradient. Next we provide a condition for a Ricci soliton on a Kaehler manifold to be a Kaehler-Ricci soliton. Finally we give an example supporting the first result.

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1. INTRODUCTION

A Ricci Soliton is a generalization of a Einstein manifold, and defined as a complete Riemannian manifold \((M,g)\) with a vector field \(V\), satisfying the equation

\[ \mathcal{L}_V g + 2\text{Ric} = 2\lambda g \]

where \(\mathcal{L}_V\) denotes Lie derivative along \(V\), \(\text{Ric}\) denotes the Ricci tensor of \(g\) and \(\lambda\) a real constant. The Ricci soliton is a special self similar solution of the Hamilton’s Ricci flow: \(\frac{\partial}{\partial t}g(t) = -2\text{Ric}(t)\) with initial condition \(g(0) = g\); and is said to be shrinking, steady, or expanding accordingly as \(\lambda > 0, = 0\) or \(< 0\) respectively. In particular, if \(V\) is the gradient of a smooth function \(f\) on \(M\), i.e., \(V = \text{grad} f\), up to the addition of a Killing vector field, then we say that the Ricci soliton is gradient and \(f\) is the potential function. For a gradient Ricci soliton, equation (1) becomes

\[ \text{Hess} f + 2\text{Ric} = 2\lambda g \]

where \(\text{Hess}\) denotes the Hessian operator \(\nabla\nabla\) (\(\nabla\) denoting the covariant derivative operator with respect to the Riemannian connection of
The following formulas are well known (see Chow et al. [1] and Petersen and Wylie [4]) for a gradient Ricci soliton:

\[ Q(\nabla f) = \frac{1}{2} \nabla S \]  
\[ |\nabla f|^2 + S - 2\lambda f = \text{a constant} \]  
\[ \Delta f - |\nabla f|^2 + 2\lambda f = \text{a constant} \]

where \( \Delta f \) is the \( g \)-trace of \( \text{Hess} f \), \( Q \) is the Ricci operator defined by \( g(QX,Y) = \text{Ric}(X,Y) \) for arbitrary vector fields \( X, Y \) on \( M \), and \( S \) denotes the scalar curvature of \( g \).

A seminal result of Perelman [3] says that a compact Ricci soliton is necessarily gradient. In this article, we first provide a geometric operator-theoretic condition on \( V \) so that it may become gradient for the non-steady case. The 1-form metrically equivalent to \( V \) is denoted by \( v \) and is given by \( v(X) = g(V,X) \) for an arbitrary vector field \( X \) on \( M \). For a \( p \)-form \( \omega \), we denote the co-differential operator by \( \delta \), i.e., \( \delta \omega \) is a \((p-1)\)-form such that \( (\delta \omega)_{i_2 \ldots i_p} = -\nabla^{i_2 \omega_{i_1 \ldots i_p}} \). The interior product operator of \( \omega \) by \( V \) is denoted by \( i_V \) such that \( (i_V \omega)_{i_2 \ldots i_p} = V^{i_1} \omega_{i_1 \ldots i_p} \).

We now state our result as follows.

**Theorem 1.1.** The Ricci soliton vector field \( V \) and its metric dual 1-form \( v \) satisfy the following intrinsic formula:

\[ 2\lambda v = d(|V|^2 + \delta v) + 2(\delta + i_V)dv \]  

So, a non-steady Ricci soliton is gradient (i.e. \( v \) is exact) if and only if \( (\delta + i_V)dv \) is exact.

**Corollary 1.2.** A non-steady Ricci soliton \((M, g, V, \lambda)\) with \( v \) closed is gradient.

**Remark 1.** In general, \( v \) closed need not imply \( v \) exact (i.e., \( V \) gradient) unless \( M \) is simply connected.

Next, we consider a Kaehler-Ricci soliton (see [1]) which is defined as a Kaehler manifold \((M, g, J)\) satisfying the Ricci soliton equation (1) for a vector field \( V \) which is an infinitesimal automorphism of the complex structure \( J \), i.e.

\[ L_V J = 0. \]

A vector field \( V \) satisfying (7) is also known as a real holomorphic vector field or a contravariant analytic vector field (see Yano [5]). It is known (see Feldman, Ilmanen and Knopf [2]) that a Ricci soliton as a Kaehler metric is a Kaehler-Ricci soliton if it is gradient. We provide a generalization of this result as follows.

**Theorem 1.3.** A Ricci soliton which is also a Kaehler metric is Kaehler-Ricci soliton if and only if \( dv \) is \( J \)-invariant.
2. Proofs of The Results

In the following $X, Y, Z$ will denote arbitrary vector fields on $M$.

**Proof of Theorem 1.1** Equation (1) can be written as

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) + 2\text{Ric}(X, Y) = 2\lambda g(X, Y)$$

(8)

The exterior derivative $dv$ of the 1-form $v$ is given by

$$g(\nabla_X V, Y) - g(\nabla_Y V, X) = 2(dv)(X, Y)$$

(9)

As $dv$ is skew-symmetric, we define a tensor field $F$ of type $(1, 1)$ by

$$(dv)(X, Y) = g(X, FY)$$

(10)

Obviously, $F$ is skew self-adjoint, i.e. $g(X, FY) = -g(Y, FX)$. Thus equation (9) assumes the form $g(\nabla_X V, Y) - g(\nabla_Y V, X) = 2g(X, FY)$. Adding it to equation (8) side by side, and factoring $Y$ out gives

$$\nabla_X V = -QX + \lambda X - FX$$

(11)

Using this equation we compute $R(Y, X)V = \nabla_Y \nabla_X V - \nabla_X \nabla_Y V - \nabla[Y, X]V$ and obtain

$$R(Y, X)V = (\nabla_X Q)Y - (\nabla_Y Q)X + (\nabla_X F)Y - (\nabla_Y F)X$$

(12)

We note that $(dv)(X, Y) = g(X, FY)$ and $dv$ is closed. Hence

$$g(X, (\nabla_Y F)Z) + g(Y, (\nabla_Z F)X) + g(Z, (\nabla_X F)Y) = 0$$

(13)

Taking inner product of (12) with $Z$ we have

$$g(R(Y, X)V, Z) = g((\nabla_X Q)Y, Z) - g((\nabla_Y Q)X, Z) + g(Z, (\nabla_X F)Y) - g(Z, (\nabla_Y F)X)$$

(14)

The skew self-adjointness of $F$ implies skew self-adjointness of $\nabla_Y F$ and so the last term of (14) including the minus sign equals $g(X, (\nabla_Y F)Z)$. Using (13) in (14) gives

$$g(R(Y, X)V, Z) = (\nabla_X \text{Ric})(Y, Z) - (\nabla_Y \text{Ric})(X, Z) - g(Y, (\nabla_Z F)X)$$

(15)

Let $(e_i)$ be a local orthonormal frame on $M$. Setting $Y = Z = e_i$ in (15) and summing over $i = 1, \ldots, n$ provides

$$\text{Ric}(X, V) = \frac{1}{2}X(S) - (\text{div} F)X$$

(16)

Next, we compute the covariant derivative of the squared $g$-norm of $V$ using (11) as follows.

$$\nabla_X |V|^2 = 2g(\nabla_X V, V) = -2\text{Ric}(X, V) + 2\lambda g(X, V) - 2g(FX, V)$$

(17)

Eliminating $\text{Ric}(X, V)$ between (16) and (17) shows

$$\nabla_X |V|^2 + X(S) = 2\lambda g(X, V) + 2((\text{div} F)X + g(FV, X))$$

(18)
In view of (10) we note that the last term in (18) is equivalent to $-2(\delta + i_V)(dv)(X)$. Hence (18) can be expressed as
\[ d(|V|^2 + S) = 2\lambda v - 2(\delta + i_V)dv \] (19)

Now, taking the $g$-trace of equation (1) gives $\delta v = S - n\lambda$ and hence we get $d\delta v = dS$. Using this consequence in (19) we obtain the formula (6). The second part of the theorem follows from this formula, because $\lambda \neq 0$ by hypothesis. This completes the proof.

**Remark 2.** Contracting the Ricci soliton equation (1) in local coordinates and then differentiating gives
\[ \nabla_j \nabla_i V^i = -\nabla_j S. \] (20)
Differentiating the Ricci soliton equation (1) gives $\nabla_i \nabla_j V^i = -\nabla_i \nabla^i V_j - \nabla_j S$. Using this and (20) we obtain
\[ R^k_j V_k + \nabla^i \nabla_i V_j = 0. \] (21)
A vector field $V$ on a Riemannian manifold $(M, g)$ satisfying equation (21) was studied by K. Yano and T. Nagano in [7] and was termed a geodesic vector field (not to be confused with vector field whose integral curves are geodesics). Actually, (21) is equivalent to the condition $(\mathcal{L}_V \nabla)(e_i, e_i) = 0$ ($i$ summed over $1, \ldots, n$), where $e_i$ is a local orthonormal frame on $M$. Obvious examples of a geodesic vector field are Killing vector fields ($\mathcal{L}_V g = 0$) and affine Killing vector fields ($\mathcal{L}_V \nabla = 0$). For a compact Riemannian manifold we know that a divergence-free geodesic vector field is Killing (see Yano [6]). We noted earlier that a Ricci soliton vector field $V$ on a Riemannian manifold (not necessarily compact) satisfies (21), and hence we conclude that a Ricci soliton vector field $V$ is a new example of a geodesic vector field in the sense of [7].

**Remark 3.** Equations (16), (19) and (6) are generalizations of the corresponding formulas (3), (4) and (5) for a gradient Ricci soliton respectively, because in the gradient case $v = df$ which implies $dv = 0$ and hence $F = 0$.

**Proof Of Theorem 1.3.** Operating $J$ on (11) we have
\[ J\nabla_X V = -JQX + \lambda JX - JFX. \]
Next, substituting $JX$ for $X$ in (11) we get
\[ \nabla_{JX} V = -QJX + \lambda JX - FXJ. \]
Taking the difference between the above two equations and noting that $J$ commutes with the Ricci operator $Q$ for a Kaehler manifold, we find
\[ J\nabla_X V - \nabla_{JX} V = (FJ - JF)X. \] (22)
At this point, we note that
\[
(\mathcal{L}_V J)X = \mathcal{L}_V JX - J\mathcal{L}_V X
= \nabla_V JX - \nabla_{JX} V - J\nabla_V X + J\nabla_X V
= J\nabla_X V - \nabla_{JX} V
\]
where we have used the fact that $J$ is parallel for a Kaehler structure.

The use of the foregoing equation in (22) gives
\[
(\mathcal{L}_V J)X = (FJ - JF)X. \tag{23}
\]
Now using the equation (11), the Kaehlerian properties:
$JQ = QJ$, $g(JX, JY) = g(X, Y)$, $g(JX, Y) = -g(X, JY)$, skew-symmetry of $F$, and a straightforward computation we obtain
\[
2[(dv)(JX, JY) - (dv)(X, Y)] = g(J(FJ - JF)X, Y).
\]
The use of (23) in the above equation provides
\[
(dv)(JX, JY) - (dv)(X, Y) = \frac{1}{2}g(J(\mathcal{L}_V J)X, Y).
\]
This shows that $\mathcal{L}_V J = 0$ if and only if $(dv)(JX, JY) = (dv)(X, Y)$, i.e. $dv$ is $J$-invariant, completing the proof.

**Remark 4.** For a gradient Ricci soliton, $v = df$ and hence $dv = 0$, and Theorem 2 implies $\mathcal{L}_V J = 0$ and so recovers the known result (mentioned earlier) that the gradient Ricci soliton on a Kaehler manifold is indeed Kaehler-Ricci soliton. Non-gradient examples satisfying the Kaehler-Ricci soliton condition $(dv)(JX, JY) = (dv)(X, Y)$ are the cases when (i) $dv = \Omega$ and (ii) $dv = \rho$ where $\Omega$ is the Kaehler 2-form defined by $\Omega(X, Y) = g(X, JY)$, and $\rho$ is the Ricci 2-form defined by $\rho(X, Y) = g(QX, JY)$. We note that, both $\Omega$ and $\rho$ are closed and $J$-invariant.

3. **An Example Supporting Theorem 1.1**

Let us consider $R^3$ with Euclidean metric $\delta_{ij}$ for which the Ricci soliton equation is
\[
\partial_i v_j + \partial_j v_i = 2\lambda \delta_{ij}.
\]
It can be verified easily that a solution of this equation is
\[
v = (\lambda x_1 + x_2 - x_3)dx_1 + (\lambda x_2 + x_3 - x_1)dx_2 + (\lambda x_3 + x_1 - x_2)dx_3. \tag{24}
\]
Computing its exterior derivative we get
\[
dv = -2(dx_1 \wedge dx_2 + dx_2 \wedge dx_3 + dx_3 \wedge dx_1). \tag{25}
\]
We also compute
\[
\delta dv = * d * (dv) = -2 * d(dx_3 + dx_1 + dx_2) = 0.
\]
\[ i_V dv = (dv)(V) = -2[(\lambda(x_3 - x_2) + 2x_1 - x_2 - x_3)dx_1 \\
+ (\lambda(x_1 - x_3) + 2x_2 - x_3 - x_1)dx_2 \\
+ (\lambda(x_2 - x_1) + 2x_3 - x_1 - x_2)dx_3]. \]

Re-arranging the terms we obtain
\[ \delta dv + i_V dv = 2\lambda[(x_2 - x_3)dx_1 + (x_3 - x_1)dx_2 + (x_1 - x_2)dx_3] \\
- d[(x_2 - x_3)^2 + (x_3 - x_1)^2 + (x_1 - x_2)^2]. \]

Let us denote the 1-form \( \delta dv + i_V dv \) by \( \theta \). It turns out that
\[ d\theta = -4\lambda(dx_1 \wedge dx_2 + dx_2 \wedge dx_3 + dx_3 \wedge dx_1). \]

Use of equation (25) in the above shows \( d\theta = 2\lambda dv \). Thus, for \( \lambda \neq 0 \), we see that \( \theta = (\delta + i_V)dv \) is not exact because \( v \) is not exact [evident from equation (25)]. This is in agreement with the conclusion of Theorem 1.1. We also note that the Ricci soliton of this example is not gradient.

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