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Bochner-Kähler and Bach flat manifolds

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Abstract

We have classified Bochner-Kähler manifolds of real dimension > 4, which are also Bach flat. In the 4-dimensional case, we have shown that, if the scalar curvature is harmonic, then it is constant. Finally, we show that the gradient of scalar curvature of any Bochner-Kähler manifold is an infinitesimal harmonic transformation, and if it is conformal then the scalar curvature is constant..

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1 Introduction

In [2] Bochner introduced a new type of curvature tensor on Kähler manifolds as an analogue of the Weyl conformal curvature tensor in Riemannian geometry. Let M be a Kähler manifold of real dimension n (which is even and equal to complex dimension $\frac{n}{2}$) with Kählerian metric g and almost complex structure J. We denote by ∇, R, Ric, Q and r the Riemannian connection, Riemannian curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature of g, respectively. The Bochner curvature tensor B of M is defined by

$$\begin{aligned} B(X,Y)Z &= R(X,Y)Z - \frac{1}{n+4}[g(Y,Z)QX - g(QX,Z)Y \\ &+ g(JY,Z)QJX - g(QJX,Z)JY + g(QY,Z)X - g(X,Z)QY \\ &+ g(QJY,Z)JX - g(JX,Z)QJY - 2g(JX,QY)JZ \\ &- 2g(JX,Y)QJZ] + \frac{r}{(n+2)(n+4)}[g(Y,Z)X - g(X,Z)Y \\ &+ g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ]. \qquad (1.1) \end{aligned}$$

where X, Y, Z denote arbitrary vector fields on M. We will follow this notation throughout this paper. A Kähler manifold M is said to be Bochner-Kähler if the Bochner tensor vanishes on M. The curvature tensor of a Riemannian metric decomposes under the orthogonal group as the sum of three irreducible parts, namely the scalar curvature, traceless Ricci tensor and the Weyl tensor. As observed by Bochner, the curvature tensor of a Kähler metric decomposes under the unitary group as the sum of three irreducible parts, namely the scalar curvature, traceless Ricci tensor and the Bochner curvature tensor. For $n > 4$, a conformally flat Kähler manifold is flat, however for $n = 4$, the conformal flatness of a Kähler metric implies only that it is Bochner-Kähler with vanishing scalar curvature. A Kähler manifold has constant holomorphic sectional curvature if and only if it is Einstein and Bochner-Kähler. It was shown by Matsumoto [11] that a Bochner-Kähler manifold of constant scalar curvature is locally symmetric. Matsumoto and Tanno [12] showed that a locally symmetric Bochner-Kähler manifold is locally isometric to either (i) a space of constant holomorphic sectional curvature, or (ii) the product of two complex spaces of constant holomorphic sectional curvatures c and $-c$. Thus, a Bochner-Kähler manifold of constant scalar curvature is locally either (i) or (ii). For details, we refer to the fundamental work of Bryant [3]. Bochner-Kähler manifolds were also studied by Chen [5], Deprez [7], Ganchev and Mihova [10], Tachibana [15], Tachibana and Liu [16], Calvaruso [4], Olszak [13], and others.

As the space of constant holomorphic sectional curvature is Einstein, we recall a generalization of Einstein metrics, called Bach flat metrics for which the Bach tensor vanishes. The notion of Bach tensor was introduced by R. Bach [1] to study conformal relativity. This is a symmetric traceless $(0, 2)$ type tensor $\mathcal{B}$ on an n-dimensional Riemannian manifold (M, g), defined as

$$\begin{aligned} \mathcal{B}(X,Y) &= \frac{1}{n-1}(\nabla_{e_i}\nabla_{e_j}W)(X,e_i,e_j,Y) \\ &+ \frac{1}{n-2}Ric(e_i,e_j)W(X,e_i,e_j,Y) \end{aligned}$$

which can also be expressed as (Chen and He [6])

$$\mathcal{B}(X,Y) = \frac{1}{n-2}[(\nabla_{e_i}C)(e_i,X)Y - g(QW(X,e_i)Y,e_i)], \qquad (1.2)$$

where i is summed over $1, 2, 3, ..., n$ and C is the $(0, 3)$-type Cotton tensor defined by

$$\begin{aligned} C(X,Y)Z &= (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) \\ &- \frac{1}{2(n-1)}[g(Y,Z)(Xr) - g(X,Z)(Yr)] \end{aligned} \tag{1.3}$$

and W is the Weyl conformal curvature tensor defined by

$$\begin{aligned} W(X,Y)Z &= R(X,Y)Z + \frac{1}{n-2}[g(QX,Z)Y - g(QY,Z)X \\ &+ g(X,Z)QY - g(Y,Z)QX \\ &- \frac{r}{(n-1)(n-2)}[g(X,Z)Y - g(Y,Z)X]. \end{aligned} \tag{1.4}$$

The Riemannian metric is said to be Bach flat if $\mathcal{B} = 0$. In view of the definition (1.2), we see that Bach flatness is a natural generalization of Einstein and conformal flatness. We also note that Bach flat metrics in dimension 4 are the critical points of the Weyl functional $\mathcal{W}(g) = \int_M |W_g|^2 dvol_g$.

The 4-dimensional half-conformally flat (i.e. self-dual or anti-self-dual) metrics are Bach flat. We also note that the Weyl curvature tensor of a Kähler metric is not simply the Bochner curvature tensor, e.g. for a Kähler manifold of real dimension $n = 4$, the Bochner curvature tensor turns out to be W^-, the anti-self-dual part of the Weyl tensor, and hence the Bochner-Kähler metrics for $n = 4$ are the same as the self-dual Kähler metrics (Bryant [3]) and thus are Bach flat, and moreover locally conformally Einstein (Derdzinski [8]). These facts motivate us to examine Bochner-Kähler manifolds that are also Bach flat. The standard example of this situation is a complex space of constant holomorphic sectional curvature because it is Bochner-Kähler as well as Einstein. We obtain the following result.

Theorem 1.1 *A Bochner-Kähler Bach flat manifold* (M, J, g) *of real dimension* $n > 4$ *is either a space of constant holomorphic sectional curvature or locally, the product of two complex spaces of constant holomorphic sectional curvatures* c *and* $-c$.

2 Preliminaries

A smooth oriented manifold M is said to be a Kähler manifold if it carries a global (1,1)-tensor field J and a Riemannian metric such that $J^2 = -I$, $g(JX, JY) = g(X, Y)$ and $\nabla J = 0$. It is orientable and has real even dimension n, and symplectic 2-form Ω defined by $\Omega(X, Y) = g(X, JY)$. The Ricci

2-form ρ defined by $\rho(X,Y) = Ric(X, JY)$ is closed. A Kähler manifold has the following properties:

$$Ric(JX, JY) = Ric(X, Y),\ QJ = JQ. \tag{2.1}$$

The covariant derivative of the Ricci tensor of a Bochner-Kähler manifold is given by

$$\begin{aligned}(2n+4)(\nabla_X Ric)(Y,Z) &= g(X,Y)(Zr) + g(X,Z)(Yr) + 2g(Y,Z)(Xr) \\ &- g(JX,Y)(JZr) - g(JX,Z)(JYr). \end{aligned} \tag{2.2}$$

We denote the gradient operator of g by D and the Laplacian of a smooth function by $\Delta = -div \circ D$. Let $(e_i)[i = 1, 2, 3, ..., n]$ be a local orthonormal frame on M. If an index i is repeated in an expression, then there is a summation over the range of i.

3 Lemmas

We now prove two Lemmas.

Lemma 3.1 *For a Bochner-Kähler manifold (M, J, g), the following formulas hold.*

$$\begin{aligned}(n+4)g(R(X,e_i)Z, Qe_i) &= -4g(Q^2X, Z) - \frac{nr}{n+2}g(QX, Z), \\ &+ (|Q|^2 - \frac{r^2}{n+2})g(X, Z), \end{aligned} \tag{3.1}$$

$$\begin{aligned}2(n+2)[g(Q^2X, Y) &- g(R(X,e_i)Z, Qe_i)] \\ &+ (\Delta r)g(X,Y) + ng(\nabla_X Dr, Y), \end{aligned} \tag{3.2}$$

$$\begin{aligned}\frac{1}{n+4}\{ng(Q^2X, Y) &- |Q|^2 g(X,Y)\} \\ &- \frac{r}{(n+2)(n+4)}\{ng(QX,Y) - rg(X,Y)\} \\ &= -\frac{1}{2(n+2)}[(\Delta r)g(X,Y) + ng(\nabla_X Dr, Y). \end{aligned} \tag{3.3}$$

Proof. By hypothesis $B = 0$. So, making use of (1.1), replacing Y by e_i, taking its inner product with Qe_i and then summing over i, together with the J-invariance (2.1) of Q one gets (3.1). Next, re-writing (2.2) as

$$\begin{aligned} 2(n+2)(\nabla_X Q)Y &= g(X,Y)Dr + g(Y,Dr)X + 2g(X,Dr)Y \\ &+ g(JX,Y)JDr - g(JY,Dr)JX, \end{aligned} \tag{3.4}$$

using it in the Ricci identity:

$$R(Z,X)QY - QR(Z,X)Y = (\nabla_Z\nabla_X Q - \nabla_X\nabla_Z Q - \nabla_{[Z,X]}Q)Y,$$

substituting e_i for Z and taking its inner product with e_i provides

$$\begin{aligned} & 2(n+2)[g(Q^2X,Y) - g(R(e_i,X)Y,Qe_i)] = \\ - & [(\Delta r)g(X,Y) + g(\nabla_{JX}Dr,JY) + (n-1)g(\nabla_X Dr,Y)] \end{aligned} \tag{3.5}$$

where we have used the equation $g(\nabla_{e_i}Dr, Je_i) = 0$ which can be obtained by taking the local orthonormal frame $\{e_i\}$ as a J-adapted frame $\{e_a, Je_a : a = 1,2,...,\frac{n}{2}\}$, and the following computation:

$$\begin{aligned} g(\nabla_{e_i}Dr, Je_i) &= g(\nabla_{e_a}Dr, Je_a) + g(\nabla_{Je_a}Dr, J^2e_a) \\ &= g(\nabla_{e_a}Dr, Je_a) - g(\nabla_{Je_a}Dr, e_a) = 0, \end{aligned}$$

through Poincaré lemma: $d^2 = 0$. Using (3.1) in (3.5) yields

$$\begin{aligned} & \frac{1}{n+4}\{ng(Q^2X,Y) - |Q|^2g(X,Y)\} - \frac{r}{(n+2)(n+4)}\{ng(QX,Y) \\ - & rg(X,Y)\} = -\frac{1}{2(n+2)}[(\Delta r)g(X,Y) \\ + & (n-1)g(\nabla_X Dr,Y) + g(\nabla_{JX}Dr,JY). \end{aligned} \tag{3.6}$$

Substituting JX and JY for X and Y respectively, in the foregoing equation, using (2.1) and then subtracting the resulting equation from (3.6) shows that

$$g(\nabla_{JX}Dr, JY) = g(\nabla_X Dr, Y). \tag{3.7}$$

Consequently, (3.6) takes the form of equation (3.3). In addition, equation (3.5) assumes the form of equation (3.2), completing the proof of Lemma 1.

Lemma 3.2 *For a Bochner-Kähler manifold, Dr is analytic, and satisfies the equation: $2Ric(X,Dr) = X(\Delta r)$.*

Though this lemma occurs in Ganchev and Mihova [10], we include its proof for the sake of completeness.

Proof. Denoting the Lie-derivative operator by L, we have $(L_{Dr}J)X = L_{Dr}JX - J(L_{Dr}X) = -\nabla_{JX} + J\nabla_X Dr$. Using this and (3.7) we find that $g((L_{Dr}J)X, JY) = 0$. Hence $L_{Dr}J = 0$, i.e. Dr is analytic. Now we recall the commutation formula (Duggal and Sharma [9], p. 39):

$$\begin{aligned}(L_{Dr}\nabla_X J - \nabla_X L_{Dr}J &- \nabla_{[Dr,X]}J)Y \\ &= (L_{Dr}\nabla)(X, JY) - J((L_{Dr}\nabla)(X, Y)).\end{aligned}$$

Using $\nabla J = 0$ and $L_{Dr}J = 0$ in the above formula, and substituting $X = e_i$, $Y = Je_i$ we get $(L_{Dr}\nabla)(e_i, e_i) + J(L_{Dr}\nabla)(e_i, e_i) = 0$. The second term vanishes, because $(L_{Dr}\nabla)(X, Y)$ and $g(X, JY)$ are symmetric and skew-symmetric respectively, in X, Y. Hence $(L_{Dr}\nabla)(e_i, e_i) = 0$, i.e.

$$\nabla_{e_i}\nabla_{e_i}Dr - \nabla_{\nabla_{e_i}e_i}Dr + R(Dr, e_i)e_i = 0$$

which is essentially, $(\nabla^i\nabla_i + Q)Dr = 0$ [$\nabla^i\nabla_i$ is the rough Laplacian in terms of a local coordinate system]. As the Hodge Laplacian acts on Dr as $(-\nabla^i\nabla_i + Q)Dr$, the preceding equation assumes the form $\Delta Dr = 2QDr$, completing the proof.

4 Proof Of Theorem 1

In order to compute the Bach tensor of the Bochner-Kähler manifold, we compute each term of the right hand side of (1.2) separately. To compute the first term, we use (2.2) in (1.3) to get

$$\begin{aligned}(2n+4)C(X, Y)Z &= -\frac{3}{n-1}\{g(Y, Z)g(X, Dr) - g(X, Z)g(Y, Dr)\} \\ &- 2g(JX, Y)g(JZ, Dr) - g(JZ, Y)g(JX, Dr) \\ &- g(JX, Z)g(JY, Dr). \qquad (4.1)\end{aligned}$$

Taking its covariant derivative along an arbitrary vector field U, and substituting e_i for U and X, and using the equation (3.7) along with the property: $g(Je_i, \nabla_{e_i}Dr) = 0$ noted earlier, we find that

$$\begin{aligned}(\nabla_{e_i}C)(e_i, Y)Z &= \frac{3\Delta r}{(2n+4)(n-1)}g(Y, Z) \\ &+ \frac{3n}{(2n+4)(n-1)}g(Y, \nabla_Z Dr). \qquad (4.2)\end{aligned}$$

Next, the use of equation (1.4) in conjuction with (3.1) shows that the second term turns out to be

$$g(QW(X,e_i)Z,e_i) = -\frac{6n}{(n+4)(n-2)}g(Q^2X,Z) + \frac{3n(3n+2)r}{(n-1)(n-2)(n+2)(n+4)}g(QX,Z) + [\frac{6|Q|^2}{(n+4)(n-2)} - \frac{3r^2(3n+2)}{(n-1)(n-2)(n+2)(n+4)}]g(X,Z).$$

At this point, we use the Bach flatness. Substituting the above expression and (4.2) in the Bach tensor (1.2) and using the hypothesis:$\mathcal{B}=0$ provides

$$\frac{3n(3n+2)r}{(n-1)(n-2)(n+2)(n+4)}g(QX,Z) - \frac{6n}{(n+4)(n-2)}g(Q^2X,Z) + [\frac{6TrQ^2}{(n+4)(n-2)} - \frac{3r^2(3n+2)}{(n-1)(n-2)(n+2)(n+4)}]g(X,Z) - \frac{3\Delta r}{(2n+4)(n-1)}g(Y,Z) - \frac{3n}{(2n+4)(n-1)}g(Y,\nabla_Z Dr) = 0.$$

Eliminating the covariant derivative of Dr between the above equation and equation (3.3) of Lemma 1, we get

$$n\{ng(Q^2X,Z) - |Q|^2 g(X,Z)\} = 2r\{ng(QX,Z) - rg(X,Z)\}. \qquad (4.3)$$

Next, eliminating Q^2 between (4.3) and (3.3) gives the equation

$$(\nabla_X Dr + \frac{\Delta r}{n}X) + \frac{2r}{n}(QX - \frac{r}{n}X) = 0 \qquad (4.4)$$

which shows that g is locally conformally Einstein, i.e. $r^{-2}g$ is Einstein over an open dense subset of M on which $r \neq 0$. Computing $R(X,Y)Dr$ through (4.4), substituting e_i for X, taking inner product with e_i, and using the twice contracted Bianchi's second identity: $divQ = \frac{1}{2}dr$ yields

$$(n+2)Ric(Y,Dr) + \frac{(n-4)}{n}r(Yr) = (n-1)(Y\Delta r). \qquad (4.5)$$

At this point, we use Lemma 2 to eliminate $Y\Delta r$ from the above equation and thus obtain

$$(n-4)[S(Y,Dr) - \frac{r}{n}g(Y,Dr)] = 0. \qquad (4.6)$$

By hypothesis, $n > 4$, and so we conclude

$$QDr = \frac{r}{n} Dr. \tag{4.7}$$

Substituting Dr for Z in (4.3) and using (4.7) provides

$$[|Q|^2 - \frac{r^2}{n}]Dr = 0. \tag{4.8}$$

Since $|Q - \frac{r}{n} I|^2 = |Q|^2 - \frac{r^2}{n}$, equation (4.8) assumes the form

$$|Q - \frac{r}{n} I|^2 Dr = 0. \tag{4.9}$$

Now, either (i) r is constant on M, or (ii) $Dr \neq 0$ on an open dense subset $\mathcal{U}$ of M. In case (ii), (4.9) implies that g is Einstein and hence r is constant, i.e. $Dr = 0$ on $\mathcal{U}$ which is a contradiction. Hence case (ii) is ruled out, and therefore r is constant on M. So, applying the results of Matsumoto [11] and Matsumoto and Tanno [12] mentioned in Section 1, we complete the proof.

5 Real 4-dimensional Case

We prove the following result.

Proposition 5.1 *If the scalar curvature of a real 4-dimensional Bochner-Kähler manifold is harmonic, then it is constant.*

Proof. As the real 4-dimensional Bochner-Kähler manifold is self-dual, it is Bach flat. For any 4-dimensional Kähler manifold we know [8] that equation (4.3) holds identically, and therefore equation (3.3) implies (4.4), i.e. g is locally conformally Einstein [8], precisely: $r^{-2}g$ is Einstein when $r \neq 0$. To prove the proposition, let us assume that r is not constant, and hence $Dr \neq 0$ on some open dense subset $\mathcal{U}$ of M. By Lemma 2, and the hypothesis $\Delta r = 0$, we have $QDr = 0$. Differentiating it along an arbitrary vector field X, using (4.4) and contracting the resulting equation with respect to X gives

$$|Dr|^2 - r|Q|^2 + \frac{r^3}{4} = 0 \tag{5.1}$$

Next, using $QDr = 0$ in equation (4.3) we get $r^2 = 2|Q|^2$. Eliminating $|Q|$ between the preceding equation and (5.1) we find

$$4|Dr|^2 = r^3 \tag{5.2}$$

At this point, we recall the following fact (Prop. 4 of [8]) for a 4-dimensional Bach flat Kähler manifold:

$$r^3 - 6r\Delta r - 12|Dr|^2$$

is constant. As $\Delta r = 0$, the aforementioned fact means that $r^3 - 12|Dr|^2$ is constant. This, in conjunction with (4.2), shows that r is constant, and hence $Dr = 0$ on $\mathcal{U}$, contradicting our assumption. This completes the proof.

Remark In the compact case, the hypothesis $\Delta r = 0$ of Proposition 1 automatically implies, by Hopf's lemma, that r is constant.

6 More On Bochner-Kähler Manifolds

We recall (Stepanov and Shelepova[14]) that a vector field V on a Riemannian manifold (M, g) is called an infinitesimal harmonic transformation if the 1-parameter group of local transformations of (M, g) generated by V consists of local harmonic diffeomorphisms, and is defined by the equation $g^{ij} L_V \Gamma^k_{ij} = 0$, where Γ^k_{ij} are the connection coefficients of g_{ij}. Such a V was called a geodesic vector field (not to be confused with a vector field whose integral curves are geodesics) by Yano and Nagano [18]. Basically, this means that V preserves the geodesics on the average, and is equivalent to the equation $\Box V = 0$, where $\Box$ is the Yano operator (Yano [17]) which is self-adjoint and acts on a smooth vector field V such that $\Box V$ is a vector field with components $-(g^{jk}\nabla_j \nabla_k V^i + R^i_j V^j)$. For a Bochner-Kähler manifold, during the proof of Lemma 3.2, we observed that Dr satisfies $\Box Dr = 0$. So, for a Bochner-Kähler manifold, the gradient of the scalar curvature $\in Ker(\Box)$,and hence is an infinitesimal harmonic transformation. We state this as follows.

Proposition 6.1 *The gradient of the scalar curvature of a Bochner-Kähler manifold lies in the kernel of the Yano operator, and hence is an infinitesimal harmonic transformation.*

Another example of an infinitesimal harmonic transformation is the associated vector field V of a Ricci soliton defined by $L_V g + 2Ric = 2\lambda g$ on a Riemannian manifold (M, g), where λ is a dilation constant [14].

For a Bochner-Kähler manifold, it is known [10] that the vector field JDr is Killing and analytic. Hence Dr is also analytic, and $[Dr, JDr] = 0$. But Dr is not necessarily Killing. If r is constant, then, evidently, the right side of equation (3.3) vanishes. The vanishing of the right side of equation

(3.3) means that Dr is conformal. We examine this condition and prove the following.

Proposition 6.2 *If the gradient of the scalar curvature r of a Bochner-Kähler manifold M is conformal, then r is constant.*

Proof. By hypothesis, $L_V g = 2\sigma g$ for a smooth function σ on M, equivalently,

$$\nabla_X Dr = \sigma X. \tag{6.1}$$

The g-trace of the above equation is

$$\Delta r = -n\sigma. \tag{6.2}$$

A straightforward computation using (6.1) and (6.2) provides

$$Ric(X, Dr) = \frac{n-1}{n} X(\Delta r).$$

This, in conjunction with Lemma 3.1, implies that Δr is constant, and so $QDr = 0$. Now, using this in (3.3) with $X = Dr$, we obtain

$$(r^2 - (n+2)|Q|^2)Dr = 0. \tag{6.3}$$

We claim that r is constant. Assume that it is not true. Then $Dr \neq 0$ on some open dense subset $\mathcal{U}$ of M. So, in view of the identity: $|Q|^2 = |Q - \frac{r}{n}I|^2 + \frac{r^2}{n}$, (6.3) implies that

$$n(n+2)|Q - \frac{r}{n}I|^2 + 2r^2 = 0$$

Thus we conclude that $Q = \frac{r}{n}I$ and $r = 0$ on $\mathcal{U}$, contradicting our assumption, and hence completing the proof.

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