2014

Almost Ricci Solitons and K-Contact Geometry

Ramesh Sharma

University of New Haven, rsharma@newhaven.edu

Follow this and additional works at: https://digitalcommons.newhaven.edu/mathematics-facpubs

Part of the Mathematics Commons

Publisher Citation
Almost Ricci solitons and K-contact geometry, Monatshefte fur Mathematik 175(4) (Dec. 2014),621-628. First published online 04 July 2014

Comments
This is the author's accepted version of the article published in Monatshefte fur Mathematik The final published article is available at Springer: http://dx.doi.org/10.1007/s00605-014-0657-8
ALMOST RICCI SOLITONS
AND K-CONTACT GEOMETRY

Ramesh Sharma
University Of New Haven, West Haven, CT 06516, USA
E-mail: rsharma@newhaven.edu

Abstract: We give a short Lie-derivative theoretic proof of the following recent result of Barros et al. “A compact non-trivial almost Ricci soliton with constant scalar curvature is gradient, and isometric to a Euclidean sphere”. Next, we obtain the result: A complete almost Ricci soliton whose metric $g$ is $K$-contact and flow vector field $X$ is contact becomes Ricci soliton with constant scalar curvature. In particular, for $X$ strict, $g$ becomes compact Sasakian Einstein. Finally, we show that the Lie-bracket of two distinct Ricci soliton vector fields with the same metric generates a steady Ricci soliton.

2010 MSC:53C25,53C44,53C21

Keywords: Almost Ricci soliton, Conformal vector field, Constant scalar curvature, $K$-contact metric, Einstein Sasakian metric.

1 Introduction

Modifying the Ricci soliton equation by allowing the dilation constant $\lambda$ to become a variable function, Pigola et al. [8] defined an almost Ricci soliton as a Riemannian manifold $(M, g)$ satisfying the condition:

$$\mathcal{L}_X g_{ij} + 2R_{ij} = 2\lambda g_{ij}. \quad (1)$$

where $X$ is a vector field on $M$, $g_{ij}$ and $R_{ij}$ are the components of the metric tensor $g$ and its Ricci tensor in local coordinates $(x^i)$, $\mathcal{L}_X$ is the Lie-derivative operator along $X$, and $\lambda$ is a smooth function on $M$. A simple example is the canonical metric $g$ on a Euclidean sphere with $X$ a non-homothetic conformal vector field. For $\lambda$ constant, (1) becomes the Ricci soliton. The
almost Ricci soliton is said to be shrinking, steady, and expanding according as \( \lambda \) is positive, zero, and negative respectively; otherwise is indefinite. If the vector field \( X \) is the gradient of a smooth function \( f \), up to the addition of a Killing vector field, \( (M,g,X,\lambda) \) is called a gradient almost Ricci soliton, in which case the equation (1) assumes the form:

\[
\nabla_i \nabla_j f + R_{ij} = \lambda g_{ij}.
\]

(2)

For an almost Ricci soliton with \( X \) homothetic, \( g \) is Einstein and hence \( \lambda \) becomes constant and it becomes the trivial Ricci soliton. For \( X \) non-homothetic, \( g \) is a non-trivial almost Ricci soliton. We also note for an almost Ricci soliton that \( X \) is conformal if and only if \( g \) is Einstein.

Ricci solitons are special solutions of the Ricci flow equation

\[
\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t),
\]

(3)
of the form \( g_{ij}(t) = \sigma(t)\psi^*_t g_{ij} \) with initial condition \( g_{ij}(0) = g_{ij} \), where \( \psi_t \) are diffeomorphisms of \( M \) and \( \sigma(t) \) is the scaling function. In the same vein, we can view almost Ricci soliton as a special solution of Ricci flow, by considering the ansatz:

\[
g_{ij}(t) = \sigma(t,x^k)\psi_t^* g_{ij},
\]

(4)

where \( \psi_t \) are diffeomorphisms of \( M \) generated by the family of vector fields \( Y(t) \), and \( \sigma(t,x^k) \) can be viewed as a pointwise scaling function that depends not only on time \( t \), but also on the coordinates \( x^k \) of points. The initial conditions: \( g_{ij}(0) = g_{ij} \), \( \psi_0 = \text{identity} \), imply \( \sigma(0,x^k) = 1 \). Differentiating (4) with respect to \( t \), using the Ricci flow equation (3), and substituting \( t = 0 \) shows

\[
-2R_{ij} = (\frac{\partial}{\partial t} \sigma(t,x^k))|_{t=0} g_{ij} + \mathcal{L}_{Y(0)} g_{ij},
\]

Labelling \( Y(0) \) as \( X \) and the time-independent function \( (\frac{\partial}{\partial t} \sigma(t,x^k))|_{t=0} \) as \(-2\lambda\), we obtain the almost Ricci soliton equation (1).

## 2 Compact Almost Ricci Soliton

It is well known that a compact Ricci soliton is gradient. This need not be true for almost Ricci soliton. In [3], Barros and Ribeiro Jr. showed that a
compact gradient almost Ricci soliton with non-trivial conformal vector field is isometric to a Euclidean sphere. Intrigued by the fact that a compact Ricci soliton with constant scalar curvature is trivial (i.e. $X$ is Killing and $g$ is Einstein), Barros, Batista and Ribeiro Jr. [2] proved the following result.

**Theorem 1 (B-B-R)** Let $(M^n, g, X, \lambda)$ be a compact oriented almost Ricci soliton. If $\text{Ric}$, $S$ and $dv_g$ denote respectively the Ricci tensor, scalar curvature and the volume form with respect to $g$, then

$$
\int_M |\text{Ric} - \frac{S}{n} g|^2 dv_g = \frac{n - 2}{2n} \int_M g(\nabla S, X) dv_g.
$$

(5)

If, in addition; $n > 2$, the almost Ricci soliton is non-trivial and the scalar curvature is constant, then $(M, g)$ is isometric to a Euclidean sphere and the almost Ricci soliton is gradient.

In this paper we provide a short Lie-derivative theoretic proof of this result, based on equations of evolution of Christoffel symbols and curvature quantities along the flow vector field $X$. We denote the Levi-Civita connection, connection coefficients, and components of curvature tensor of $g$ by $\nabla$, $\Gamma^i_{jk}$, and $R^h_{kji}$ respectively.

**Another Proof Of Theorem 1 (B-B-R).** Let us denote the inverse of $g_{ij}$ by $g^{ij}$. Taking the Lie-derivative of the relation $g^{ij} g_{jk} = \delta^i_k$ along $X$, using equation (1) and subsequently operating the resulting equation by $g^{il}$ we immediately get

$$
\mathcal{L}_X g^{kl} = 2 R^{kl} - 2 \lambda g^{kl}.
$$

(6)

Next, the use of equation (1) in the formula (page 23, Yano [9]):

$$
\mathcal{L}_X \Gamma^h_{ij} = \frac{1}{2} g^{ht} [\nabla_j(\mathcal{L}_X g_{it}) + \nabla_i(\mathcal{L}_X g_{jt}) - \nabla_t(\mathcal{L}_X g_{ij})],
$$

yields the evolution equation

$$
\mathcal{L}_X \Gamma^h_{ij} = \nabla^h R_{ij} - \nabla_j R^h_i - \nabla_i R^h_j - (\nabla^h \lambda) g_{ij} + (\nabla_j \lambda) \delta^h_i + (\nabla_i \lambda) \delta^h_j.
$$

(7)

Let us follow the notational convention: $\nabla_k \nabla_j Z^h - \nabla_j \nabla_k Z^h = R^h_{kji} Z^i$, where $Z^i$ are components of an arbitrary vector field, and $R^h_{kji} = R_{kji}$. Using equation (7) in the following commutation formula (page 23, [9]):

$$
\nabla_k(\mathcal{L}_X \Gamma^h_{ij}) - \nabla_j(\mathcal{L}_X \Gamma^h_{ik}) = \mathcal{L}_X R^h_{kji},
$$
we obtain the evolution equation:

\[ \mathcal{L}_X R_{kji}^h = \nabla_j \nabla_k R_{i}^h - \nabla_k \nabla_j R_{i}^h + \nabla_j \nabla_i R_{k}^h - \nabla_k \nabla_i R_{j}^h + \nabla_k \nabla^h R_{ij} - \nabla_j \nabla^h R_{ik} + (\nabla_k \nabla_i \lambda) \delta_j^h - (\nabla_j \nabla^h \lambda) \delta_k^h + (\nabla_j \nabla^h \lambda) g_{ik}. \]

Contracting this equation with \( g^{hk} \) and using the twice contracted Bianchi identity: \( \nabla_i R^i_{j} = \frac{1}{2} \nabla_j S \), we have

\[ \mathcal{L}_X R_{ji} = \nabla_j \nabla_i S - \nabla_h \nabla_j R_{i}^h - \nabla_h \nabla_i R_{j}^h + \Delta R_{ij} - (\Delta \lambda) g_{ij} - (n - 2) \nabla_i \nabla_j \lambda. \]

Lie-differentiating \( S = R_{ij} g^{ij} \) along \( X \), and using the above equation and equation (6) provides the evolution equation for the scalar curvature:

\[ \mathcal{L}_X S = 2 R_{ij} R^{ij} + \Delta S - 2 \lambda S - 2(n - 1) \Delta \lambda. \] (8)

Writing \( \mathcal{L}_X S \) as \( g(\nabla S, X) \), integrating the above equation over the compact \( M \) and using the Gauss divergence theorem we get

\[ \int_M [R_{ij} R^{ij} - \lambda S - \frac{1}{2} g(\nabla S, X)] dv_g = 0. \] (9)

At this point, we note

\[ div(SX) = \nabla_i (SX^i) = g(\nabla S, X) + S \text{div} X, \]

and integrate it over \( M \) in order to get

\[ \int_M [g(\nabla S, X) + S \text{div} X] dv_g = 0. \] (10)

Now we contract equation (1) with \( g^{ij} \) in order to get \( div X = n \lambda - S \), and use it in (10) to obtain

\[ \int_M (n \lambda S - S^2 + g(\nabla S, X)) dv_g = 0. \]

Eliminating \( \int_M (\lambda S) dv_g \) between the above equation and (9) and noting \( |\mathcal{R} - \frac{S}{n} g|^2 = R_{ij} R^{ij} - \frac{S^2}{n} \) we obtain equation (5), proving the first part
of the theorem.

For the second part, we use the hypothesis that $S$ is constant in equation (5) to conclude that $g$ is Einstein. Thus, equation (1) reduces to $\mathcal{L}_X g_{ij} = 2(\lambda - \frac{S}{n})g_{ij}$, i.e. $X$ is a non-homothetic conformal vector field on $M$. With the setting $\lambda - \frac{S}{n} = \rho$, the foregoing conformal equation assumes the form

$$\mathcal{L}_X g_{ij} = 2\rho g_{ij}. \quad (11)$$

Using the conformal integrability condition (p. 26, [9])

$$\mathcal{L}_X R_{ij} = (2 - n)\nabla_i \nabla_j \rho - (\Delta \rho)g_{ij}$$

and the Einstein condition $R_{ij} = \frac{S}{n}g_{ij}$ we get

$$(\Delta \rho + \frac{2S}{n} \rho)g_{ij} = (2 - n)\nabla_i \nabla_j \rho. \quad (12)$$

Contracting it with $g^{ij}$ gives $\Delta \rho = -\frac{S}{n-1}\rho$. Using this in the identity: $\Delta \rho^2 = \nabla^i \nabla_i (\rho^2) = 2||\nabla \rho||^2 + \rho \Delta \rho$, and integrating over $M$ gives $\int_M ||\nabla \rho||^2 = \frac{S}{n-1} \int_M \rho^2$. This shows that $S > 0$. Consequently, equation (12) becomes

$$\nabla_i \nabla_j \rho = -\frac{S}{n(n-1)} \rho g_{ij}. \quad (13)$$

This implies, by virtue of Obata's theorem [7]: “A complete Riemannian manifold $(M,g)$ of dimension $n \geq 2$ admits a non-trivial solution $\rho$ of the system of partial differential equations $\nabla_i \nabla_j \rho = -c^2 \rho g_{ij}$ ($c$ a positive constant) if and only if $M$ is isometric to a Euclidean sphere of radius $1/c$” that $(M,g)$ is isometric to a Euclidean sphere of radius $\sqrt{\frac{n(n-1)}{S}}$.

Equation (13) can also be expressed as $\mathcal{L}_{\nabla \rho} g_{ij} = \frac{2S}{n(1-n)} \rho g_{ij}$. Combining this with (11) we obtain

$$\mathcal{L}_X \frac{1}{n} \nabla \rho g_{ij} = 0.$$

Hence $X = \nabla (\frac{1}{n} \nabla \rho)$ + a Killing vector field, i.e. the almost Ricci soliton is gradient, completing the proof.
3 $K$-Contact Metric As Almost Ricci Soliton

A $(2m + 1)$-dimensional smooth manifold $M$ is called a contact manifold if it carries a global 1-form $\eta$ such that $\eta \wedge (d\eta)^m \neq 0$ everywhere on $M$. For a given contact 1-form $\eta$ there exists a unique vector field $\xi$ (Reeb vector field) such that $(d\eta)(\xi,.) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, one obtains a Riemannian metric $g$ and a $(1,1)$-tensor field $\varphi$ such that

$$(d\eta)(Y,Z) = g(Y,\varphi Z), \eta(Z) = g(\xi, Z), \varphi^2 = -I + \eta \otimes \xi,$$  \hspace{1cm} (14)$$

for arbitrary vector fields $Y, Z$ on $M$. We call $g$ an associated metric of $\eta$ and $(\varphi, \eta, \xi, g)$ a contact metric structure. A $K$-contact metric is a contact metric for which $\xi$ is Killing, equivalently:

$$Ric(\xi, Y) = 2mg(\xi, Y),$$  \hspace{1cm} (15)$$

for an arbitrary vector field $Y$ on $M$. This condition is also equivalent to:

$$Ric(\xi, \xi) = 2m.$$  \hspace{1cm} (16)$$

For details we refer to [4]. A contact metric $g$ on $M^{2m+1}$ is called Sasakian if the almost Kaehler structure induced on the cone $(\mathbb{R}^+ \times M)$ with metric $dr^2 + r^2 g$, is Kaehler (see Boyer and Galicki [5]). A Sasakian metric is $K$-contact, but the converse need not be true, except in dimension 3.

We would like to consider an almost Ricci soliton $(M, g, X, \lambda)$ such that $g$ is a $K$-contact metric and $X$ is a contact vector field. Let us recall that a vector field $X$ on a contact manifold is said to be a contact vector field if

$$\mathcal{L}_X \eta = f \eta,$$  \hspace{1cm} (17)$$

for a smooth function $f$ on $M$. The contact vector field $X$ is called strict when $f = 0$.

Using Cartan’s magic formula, we find that $\mathcal{L}_\xi \eta = di_\xi \eta + i_\xi d\eta = d(1) + d\eta(\xi,.) = 0$, i.e. $\xi$ is a strict contact vector field. We note from equation (1) that, if we take $g$ as a $K$-contact metric and $X$ as $\xi$, then (as $\xi$ is Killing), the $K$-contact metric $g$ reduces to an Einstein metric and $\lambda$ becomes constant, equal to the Einstein constant $2m$, as seen from equation (16). We generalize this special situation in the form of the following result.
Theorem 2 Let \((M, g, X, \lambda)\) be a complete almost Ricci soliton with \(g\) a K-contact metric and \(X\) a contact vector field. Then it becomes Ricci soliton and \(g\) has constant scalar curvature. In particular, if \(X\) is strict, then \(g\) is Sasakian Einstein.

Proof. First of all, we have, by definition of the contact structure, \(\omega = \eta \wedge (d\eta)^m \neq 0\) and thus is a volume element. Denote it by \(\omega\). Using the hypothesis (17) we compute \(\mathcal{L}_X d\eta = d\mathcal{L}_X \eta = (df) \wedge \eta + f(d\eta)\). Consequently, the formula: \(\mathcal{L}_X \omega = (\text{div}X) \omega\) yields the relation \(\text{div}X = (m + 1)f\). On the other hand, the trace of equation (1) is \(\text{div}X = (2m + 1)(m + 1) - S\). Comparing the two values of \(\text{div}X\) we have

\[S = (2m + 1)\lambda - (m + 1)f.\]  

(18)

Next, we Lie-differentiate the second equation in (14) along \(X\), and then use equations (1), (15) and (17) in order to get

\[\mathcal{L}_X \xi = (f - 2\lambda + 4m)\xi.\]  

(19)

The Lie-derivative of \(g(\xi, \xi) = 1\) (as \(\xi\) is unit) along \(X\), and the use of equations (1) and (16) provides \(g(\mathcal{L}_X \xi, \xi) = 2m - \lambda\). The inner product of (19) with \(\xi\) and the foregoing equation lead us to the relation: \(f = \lambda - 2m\). Consequently, we have

\[\mathcal{L}_X \eta = (\lambda - 2m)\eta, \quad \mathcal{L}_X \xi = (2m - \lambda)\xi.\]  

(20)

At this point, we take the Lie-derivative of the first equation in (14), along \(X\) and use equations (1) and (17) in order to obtain

\[\eta(Z) \nabla f - (Zf)\xi + 2(f - 2\lambda)\varphi Z = -4Q\varphi Z + 2(\mathcal{L}_X \varphi)Z,\]  

(21)

where \(Z\) is an arbitrary vector field on \(M\), and \(Q\) is the Ricci tensor of type (1,1), defined by \(g(Q\ldots, \ldots) = \text{Ric}(\ldots)\). Substituting \(\xi\) for \(Z\) in equation (21) and using the property \(\varphi\xi = 0\) and equation (20) we find \(\nabla f = (\xi f)\xi\), i.e. \(df = (\xi f)\eta\). Taking its exterior derivative, using Poincaré lemma: \(d^2 = 0\), and then wedge product with \(\eta\) we have \((\xi f)\eta \wedge d\eta = 0\). As \(\eta \wedge d\eta\) cannot vanish anywhere, otherwise the definition of the contact structure would be violated, we conclude that \(\xi f = 0\), and hence \(df = 0\), i.e. \(f\) is constant on \(M\). Consequently, equation (21) reduces to the following evolution equation for \(\varphi\):

\[\mathcal{L}_X \varphi = 2Q\varphi - (2m + \lambda)\varphi.\]  

(22)
As shown earlier, \( f = \lambda - 2m \), and \( f \) is constant, we conclude that \( \lambda \) is constant and hence the almost Ricci soliton becomes Ricci soliton. Appealing to equation (18), we find that \( S \) is constant. This proves first part. For the second part, the hypothesis \( f = 0 \) immediately implies \( \lambda = 2m \) and thus we get from (18) that \( S = 2m(2m + 1) \). Plugging these findings in equation (8) and carrying out a straightforward computation shows \( |Ric - 2mg|^2 = 0 \). Hence \( Ric = 2mg \), i.e. \( g \) is Einstein with Einstein constant \( 2m \).

As \( (M, g) \) is complete, thanks to Myers’ theorem, \( (M, g) \) becomes compact. In order to turn \( g \) into Sasakian, we recall the following result of Morimoto [6]: “Let \((M, \eta, g)\) be a compact \( K \)-contact manifold such that \( g \) is \( \eta \)-Einstein, i.e. its Ricci tensor satisfies \( Ric = ag + b\eta \otimes \eta \) for real constants \( a, b \). If \( a > -2 \), then \( g \) is Sasakian”. This result was also proved independently by Boyer and Galicki [5], and Apostolov et al. [1]. In our case, \( a = 2m \) and \( b = 0 \), and hence the aforementioned result holds. Thus, we conclude that \( g \) is Sasakian, and complete the proof.

4 Commutation Of Ricci Soliton Vector Fields

We consider two distinct Ricci solitons with the same Riemannian metric and show that the Lie-bracket of their flow vector fields give rise to a steady Ricci soliton. More precisely, we prove

**Proposition 1** Let \((M, g, X_1, \lambda_1)\) and \((M, g, X_2, \lambda_2)\) be two distinct non-trivial Ricci solitons. Then, \([X_1, X_2]\) determines a steady Ricci soliton on \( M \) with a metric homothetic to \( g \).

**Proof** By hypothesis, we have

\[
\mathcal{L}_{X_1} g + 2Ric = 2\lambda_1 g, \quad \mathcal{L}_{X_2} g + 2Ric = 2\lambda_2 g,
\]

where \( \lambda_1 \) and \( \lambda_2 \) are constants. As these are two distinct Ricci solitons, we may assume without any loss of generality, that \( \lambda_1 < \lambda_2 \). The two equations in (23) show that \( X_1 = X_2 + H \) where \( H \) is a homothetic vector field satisfying \( \mathcal{L}_H g = 2(\lambda_1 - \lambda_2)g \). The following computation:

\[
\mathcal{L}_{[X_1, X_2]} g = \mathcal{L}_{[X_2 + H, X_2]} g = \mathcal{L}_{[H, X_2]} g = \mathcal{L}_H \mathcal{L}_{X_2} g - \mathcal{L}_{X_2} \mathcal{L}_H g
\]

\[
= \mathcal{L}_H(-2Ric + 2\lambda_2 g) - \mathcal{L}_{X_2}(2(\lambda_1 - \lambda_2)g) = 4(\lambda_1 - \lambda_2)Ric,
\]

8
shows that
\[ \mathcal{L}_{\frac{1}{2(\lambda_2 - \lambda_1)}[X_1, X_2]}g + 2Ric = 0. \]

Taking into account the fact that the Ricci tensor is invariant under a homothetic transformation and noticing that \([X_1, X_2]\) cannot be conformal (otherwise \(g\) would become Einstein) we conclude that \((M, \frac{1}{2(\lambda_2 - \lambda_1)}g, [X_1, X_2], 0)\) is a Ricci soliton which is steady. This completes the proof.

5 Concluding Remarks

1. In the proof of Theorem 1, Barros, Batista and Ribiero Jr. used a result of Yano and Nagano and the Hodge-de Rham decomposition. Our proof uses a theorem of Obata and does not need Hodge-de Rham decomposition.

2. The hypotheses of Theorem 2 can be interpreted in terms of contact Hamiltonians as follows. The contact Hamiltonian associated to a contact vector field \(X\) defined by equation (17) is a function \(\mathcal{H}\) defined as \(\eta(X)\), and the function \(f\) turns out to be equal to \(\xi \mathcal{H}\). The vector field \(X\) is the Hamiltonian vector field associated to \(\mathcal{H}\). Computing \(\mathcal{L}_X \mathcal{H} = \mathcal{L}_X (\eta(X)) = (\mathcal{L}_X \eta)X = f \eta(X) = f \mathcal{H} = (\xi \mathcal{H}) \mathcal{H}\) shows that the contact vector field \(X\) is strict, i.e. \(f = \xi \mathcal{H} = 0\) if and only if the associated Hamiltonian \(\mathcal{H}\) is a first integral of \(X\), i.e. is preserved along the flow of the Hamiltonian vector field \(X\).

3. For the second part of Theorem 2, we found that \(\lambda = 2m\), \(Ric = 2mg\) and hence \(Q = 2mI\). Using these and the hypothesis \(f = 0\) in equations (20) and (22) we infer that \(X\) preserves all structure tensors \(\eta, \xi, g, \varphi\), and hence is an infinitesimal automorphism of the Sasakian structure on \(M\).

Acknowledgments: The author thanks Dr. Amalendu Ghosh for help on a couple of points. This work has been supported by University Research Scholarship of the University of New Haven.

References


