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Sasakian metric as a Ricci soliton and related results

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Abstract: We prove the following results: (i) A Sasakian metric as a non-trivial Ricci soliton is null \(\eta\)-Einstein, and expanding. Such a characterization permits to identify the Sasakian metric on the Heisenberg group \(\mathcal{H}^{2n+1}\) as an explicit example of (non-trivial) Ricci soliton of such type. (ii) If an \(\eta\)-Einstein contact metric manifold \(M\) has a vector field \(V\) leaving the structure tensor and the scalar curvature invariant, then either \(V\) is an infinitesimal automorphism, or \(M\) is \(D\)-homothetically fixed \(K\)-contact.

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1 Introduction

A Ricci soliton is a natural generalization of an Einstein metric, and is defined on a Riemannian manifold \((M, g)\) by

\[
(\mathcal{L}_V g)(X, Y) + 2Ric(X, Y) + 2\lambda g(X, Y) = 0
\]

where \(\mathcal{L}_V g\) denotes the Lie derivative of \(g\) along a vector field \(V\), \(\lambda\) a constant, and arbitrary vector fields \(X, Y\) on \(M\). The Ricci soliton is said to be shrinking, steady, and expanding accordingly as \(\lambda\) is negative, zero, and positive respectively. Actually, a Ricci soliton is a generalized fixed point of Hamilton’s Ricci flow \([7]\): \(\frac{\partial}{\partial t} g_{ij} = -2R_{ij}\), viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. For
details, see Chow et al. [4]. The vector field $V$ generates the Ricci soliton viewed as a special solution of the Ricci flow. A Ricci soliton is said to be a gradient Ricci soliton, if $V = -\nabla f$ (up to a Killing vector field) for a smooth function $f$. Ricci solitons are also of interest to physicists who refer to them as quasi-Einstein metrics (for example, see Friedan [6]).

An odd dimensional analogue of Kaehler geometry is the Sasakian geometry. The Kaehler cone over a Sasakian Einstein manifold is a Calabi-Yau manifold which has application in physics in superstring theory based on a 10-dimensional manifold that is the product of the 4-dimensional space-time and a 6-dimensional Ricci-flat Kaehler (Calabi-Yau) manifold (see Candelas et al. [3]). Sasakian geometry has been extensively studied since its recently perceived relevance in string theory. Sasakian Einstein metrics have received a lot of attention in physics, for example, $p$-brane solutions in superstring theory, Maldacena conjecture (AdS/CFS duality) [9]. For details, see Boyer, Galicki and Matzeu [2].

In [12] Sharma showed that if a $K$-contact (in particular, Sasakian) metric is a gradient Ricci soliton, then it is Einstein. This was also shown later independently by He and Zhu [8] for the Sasakian case. Recently, Sharma and Ghosh [13] proved that a 3-dimensional Sasakian metric which is a non-trivial (i.e. non-Einstein) Ricci soliton, is homothetic to the standard Sasakian metric on $\text{nil}^3$. In this paper, we generalize these results and also answer the following question of H.-D. Cao (cited in [8]): “Does there exist a shrinking Ricci soliton on a Sasakian manifold, which is not Einstein?” by proving

**Theorem 1** If the metric of a $(2n + 1)$-dimensional Sasakian manifold $M$ ($\eta, \xi, g, \varphi$) is a non-trivial (non-Einstein) Ricci soliton, then (i) $M$ is null $\eta$-Einstein (i.e. $D$-homothetically fixed and transverse Calabi-Yau), (ii) the Ricci soliton is expanding, and (iii) the generating vector field $V$ leaves the structure tensor $\varphi$ invariant, and is an infinitesimal contact $D$-homothetic transformation.

Conversely, we consider the following question: “What can we say about an $\eta$-Einstein contact metric manifold $M$ which admits a vector field $V$ that leaves $\varphi$ invariant?” and answer it by assuming the invariance of the scalar curvature under $V$, in the form of the following result.

**Theorem 2** If an $\eta$-Einstein contact metric manifold $M$ admits a vector field $V$ that leaves the structure tensor $\varphi$ and the scalar curvature invariant,
then either \( V \) is an infinitesimal automorphism, or \( M \) is \( D \)-homothetically fixed and \( K \)-contact.

**Remark 1** Note that a Ricci soliton as a Sasakian metric is different from the Sasaki-Ricci soliton in the context of transverse Kaehler structure in a Sasakian manifold, for example see Futaki et al. [5]).

**Remark 2** Boyer et al. [2] have studied \( \eta \)-Einstein geometry as a class of distinguished Riemannian metrics on contact metric manifolds, and proved the existence of \( \eta \)-Einstein metrics on many different compact manifolds. We would also like to point out that Zhang [18] showed that compact Sasakian manifolds with constant scalar curvature and satisfying certain positive curvature condition is \( \eta \)-Einstein.

**Remark 3** Theorem 2 provides a generalization of the infinitesimal version of the following result of Tanno [15] “The group of all diffeomorphisms \( \Phi \) which leave the structure tensor \( \varphi \) of a contact metric manifold \( M \) invariant, is a Lie transformation group, and coincides with the automorphism group \( A \) if \( M \) is Einstein.” Note that the scalar curvature of an Einstein metric is constant. We also note that the set of all vector fields on a contact metric manifold \( M \), that leave \( \varphi \) and scalar curvature invariant, forms a Lie subalgebra of the Lie algebra of all smooth vector fields on \( M \).

### 2 A Brief Review Of Contact Geometry

A \((2n + 1)\)-dimensional smooth manifold is said to be contact if it has a global 1-form \( \eta \) such that \( \eta \wedge (d\eta)^n \neq 0 \) on \( M \). For a contact 1-form \( \eta \) there exists a unique vector field \( \xi \) such that \( d\eta(\xi, X) = 0 \) and \( \eta(\xi) = 1 \). Polarizing \( d\eta \) on the contact subbundle \( \eta = 0 \), we obtain a Riemannian metric \( g \) and a \((1,1)\)-tensor field \( \varphi \) such that

\[
d\eta(X, Y) = g(X, \varphi Y), \eta(X) = g(X, \xi), \varphi^2 = -I + \eta \otimes \xi
g\quad (2)
\]

\( g \) is called an associated metric of \( \eta \) and \((\varphi, \eta, \xi, g)\) a contact metric structure. Following [1] we recall two self-adjoint operators \( h = \frac{1}{2} L_\xi \varphi \) and \( l = R(., \xi)\xi \). The tensors \( h, h\varphi \) are trace-free and \( h\varphi = -\varphi h \). We also have these formulas for a contact metric manifold.

\[
\nabla_X \xi = -\varphi X - \varphi h X \quad (3)
\]
\[ l - \varphi l \varphi = -2(h^2 + \varphi^2) \]  
(4)

\[ \nabla_{\xi} h = \varphi - \varphi l - \varphi h^2 \]  
(5)

\[ Tr l = Ric(\xi, \xi) = 2n - Tr h^2 \]  
(6)

where \( \nabla, R, Ric \) and \( Q \) denote respectively, the Riemannian connection, curvature tensor, Ricci tensor and Ricci operator of \( g \). For details see [1].

A vector field \( V \) on a contact metric manifold \( M \) is said to be an infinitesimal contact transformation if \( \mathcal{L}_V \eta = \sigma \eta \) for some smooth function \( \sigma \) on \( M \). \( V \) is said to be an infinitesimal automorphism of the contact metric structure if it leaves all the structure tensors \( \eta, \xi, g, \varphi \) invariant (see Tanno [14]).

A contact metric structure is said to be \( K \)-contact if \( \xi \) is Killing with respect to \( g \), equivalently, \( h = 0 \). The contact metric structure on \( M \) is said to be Sasakian if the almost Kaehler structure on the cone manifold \( (M \times R^+, r^2 g + dr^2) \) over \( M \), is Kaehler. Sasakian manifolds are \( K \)-contact and \( K \)-contact 3-manifolds are Sasakian. For a Sasakian manifold,

\[ (\nabla_X \varphi) Y = g(X, Y) \xi - \eta(Y) X \]  
(7)

\[ R(X, Y) \xi = \eta(Y) X - \eta(X) Y, \quad Q \xi = 2n \xi \]  
(8)

For a Sasakian manifold, the restriction of \( \varphi \) to the contact sub-bundle \( D (\eta = 0) \) is denoted by \( J \) and \( (D, J, d\eta) \) defines a Kaehler metric on \( D \), with the transverse Kaehler metric \( g^T \) related to the Sasakian metric \( g \) as \( g = g^T + \eta \otimes \eta \). One finds by a direct computation that the transverse Ricci tensor \( Ric^T \) of \( g^T \) is given by

\[ Ric^T(X, Y) = Ric(X, Y) + 2g(X, Y) \]

for arbitrary vector fields \( X, Y \) in \( D \). The Ricci form \( \rho \) and transverse Ricci form \( \rho^T \) are defined by

\[ \rho(X, Y) = Ric(X, \varphi Y), \quad \rho^T(X, Y) = Ric^T(X, \varphi Y) \]

for \( X, Y \in D \). The basic first Chern class \( 2\pi c_1^B \) of \( D \) is represented by \( \rho^T \). In case \( c_1^B = 0 \), the Sasakian structure is said to be null (transverse Calabi-Yau). We refer to [2] for details.
A contact metric manifold $\mathcal{M}$ is said to be $\eta$-Einstein in the wider sense, if the Ricci tensor can be written as

\[ Ric(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y) \tag{9} \]

for some smooth functions $\alpha$ and $\beta$ on $\mathcal{M}$. It is well-known (Yano and Kon [17]) that $\alpha$ and $\beta$ are constant if $\mathcal{M}$ is $K$-contact, and has dimension greater than 3.

Given a contact metric structure $(\eta, \xi, g, \varphi)$, let $\bar{\eta} = a\eta$, $\bar{\xi} = \frac{1}{a}\xi$, $\bar{\varphi} = \varphi$, $\bar{g} = ag + a(a - 1)\eta \otimes \eta$ for a positive constant $a$. Then $(\bar{\eta}, \bar{\xi}, \bar{\varphi}, \bar{g})$ is again a contact metric structure. Such a change of structure is called a $D$-homothetic deformation, and preserves many basic properties like being $K$-contact (in particular, Sasakian). It is straightforward to verify that, under a $D$-homothetic deformation, a $K$-contact $\eta$-Einstein manifold transforms to a $K$-contact $\eta$-Einstein manifold such that $\bar{\alpha} = \frac{\alpha + 2}{a} - \frac{2\alpha}{a}$ and $\bar{\beta} = 2n - \alpha$.

We remark here that the particular value: $\alpha = -2$ remains fixed under a $D$-homothetic deformation, and as $\alpha + \beta = 2n$, $\beta$ also remains fixed. Thus, we state the following definition.

**Definition 1** A $K$-contact $\eta$-Einstein manifold with $\alpha = -2$ is said to be $D$-homothetically fixed.

### 3 Proofs Of The Results

**Proof Of Theorem 1:** Using the Ricci soliton equation (1) in the commutation formula (Yano [16], p.23)

\[
(\mathcal{L}_V \nabla_X g - \nabla_X \mathcal{L}_V g - \nabla_{[V,X]} g)(Y, Z) =
- g((\mathcal{L}_V \nabla)(X, Y), Z) - g((\mathcal{L}_V \nabla)(X, Z), Y) \tag{10}
\]

we derive

\[
g((\mathcal{L}_V \nabla)(X, Y), Z) = (\nabla_Z Ric)(X, Y)
- (\nabla_X Ric)(Y, Z) - (\nabla_Y Ric)(X, Z) \tag{11}
\]

As $\xi$ is Killing, we have $\mathcal{L}_\xi Ric = 0$ which, in view of (3), the last equation of (8) and $h = 0$, is equivalent to $\nabla_\xi Q = Q\varphi - \varphi Q$. But for a Sasakian
manifold, \( Q \) commutes with \( \varphi \), and hence \( Ric \) is parallel along \( \xi \). Moreover, differentiating the last equation of (8), we have \( (\nabla_X Q)\xi = Q\varphi X - 2n\varphi X \). Substituting \( \xi \) for \( Y \) in (11) and using these consequences we obtain

\[
(\mathcal{L}_V \nabla)(X, \xi) = -2Q\varphi X + 4n\varphi X \tag{12}
\]

Differentiating this along an arbitrary vector field \( Y \), using (7) and the last equation of (8), we find

\[
(\nabla_Y \mathcal{L}_V \nabla)(X, \xi) - (\mathcal{L}_V \nabla)(X, \varphi Y) = -2(\nabla_Y Q)\varphi X + 2\eta(X)QY - 4n\eta(X)Y
\]

The use of the foregoing equation in the commutation formula [16]:

\[
(\mathcal{L}_V R)(X, Y)Z = (\nabla_X \mathcal{L}_V \nabla)(Y, Z) - (\nabla_Y \mathcal{L}_V \nabla)(X, Z) \tag{13}
\]

for a Riemannian manifold, shows that

\[
(\mathcal{L}_V R)(X, Y)\xi - (\mathcal{L}_V \nabla)(Y, \varphi X) + (\mathcal{L}_V \nabla)(X, \varphi Y) = -2(\nabla_X Q)\varphi Y + 2(\nabla_Y Q)\varphi X + 2\eta(Y)QX - 2\eta(X)QY + 4n\eta(X)Y - 4n\eta(Y)X
\]

Substituting \( \xi \) for \( Y \) in the foregoing equation, using (12) and the formula \( \nabla_\xi Q = 0 \) noted earlier, we find that

\[
(\mathcal{L}_V R)(X, \xi)\xi = 4(QX - 2nX) \tag{14}
\]

Equation (1) gives \( (\mathcal{L}_V g)(X, \xi) + 2(2n + \lambda)\eta(X) = 0 \), which in turn, gives

\[
(\mathcal{L}_V \eta)(X) - g(\mathcal{L}_V \xi, X) + 2(\lambda + 2n)\eta(X) = 0 \tag{15}
\]

\[
\eta(\mathcal{L}_V \xi) = (2n + \lambda) \tag{16}
\]

where we used the Lie-derivative of \( g(\xi, \xi) = 1 \) along \( V \). Next, Lie-differentiating the formula \( R(X, \xi)\xi = X - \eta(X)\xi \) [a consequence of the first formula in (8)] along \( V \), and using equations (14) and (16) provides

\[
4(QX - 2nX) - g(\mathcal{L}_V \xi, X)\xi + 2(2n + \lambda)X = -((\mathcal{L}_V \eta)(X))\xi
\]

By the direct application of (15) to the the above equation we find

\[
Ric(X, Y) = (n - \frac{\lambda}{2})g(X, Y) + (n + \frac{\lambda}{2})\eta(X)\eta(Y) \tag{17}
\]
which shows that $M$ is $\eta$-Einstein with scalar curvature

$$r = 2n(n+1) - n\lambda$$ (18)

At this point, we recall the following integrability formula [12]:

$$\mathcal{L}_V r = -\Delta r + 2\lambda r + 2|Q|^2$$ (19)

for a Ricci soliton, where $\Delta r = -\text{div} Dr$. A straightforward computation using (17) gives the squared norm of the Ricci operator as $|Q|^2 = 2n(n^2 - n\lambda + \frac{\lambda^2}{4} + 4n^2)$. Using this and (18) in (19), we obtain the quadratic equation $(2n + \lambda)(2n + 4 - \lambda) = 0$. As $\lambda = -2n$ corresponds to $g$ becoming Einstein, we must have $\lambda = 2n + 4$ and hence the soliton is expanding, which proves part (ii). Moreover, equation (18) reduces to $r = -2n$. Thus equation (17) assumes the form

$$\text{Ric}(Y, Z) = -2g(Y, Z) + 2(n+1)\eta(Y)\eta(Z)$$ (20)

Hence, as defined in Section 2, $M$ is a $D$-homothetically fixed null $\eta$-Einstein manifold, proving part (i). Using (20) in (11) provides

$$(\mathcal{L}_V \nabla)(Y, Z) = 4(n+1)\{\eta(Y)\varphi Z + \eta(Z)\varphi Y\}$$ (21)

Differentiating this along $X$, using equations (3) and (7), incorporating the resulting equation in (13), and finally contracting at $X$ we get

$$(\mathcal{L}_V \text{Ric})(Y, Z) = 8(n+1)\{g(Y, Z) - (2n+1)\eta(Y)\eta(Z)\}$$ (22)

Equation (20) reduces the soliton equation (1) to the form

$$(\mathcal{L}_V g)(Y, Z) = -4(n+1)\{g(Y, Z) + \eta(Y)\eta(Z)\}$$ (23)

Next, Lie-differentiating (20) along $V$, and using (23) shows

$$(\mathcal{L}_V \text{Ric})(Y, Z) = 8(n+1)\{g(Y, Z) + \eta(Y)\eta(Z)\}
+ 2(n+1)\{\eta(Z)(\mathcal{L}_V \eta)(Y) + \eta(Y)(\mathcal{L}_V \eta)Z\}$$ (24)

Comparing equations (22) with (24) and substituting $\xi$ for $Z$ leads to

$$\mathcal{L}_V \eta = -4(n+1)\eta$$ (25)
Therefore, substituting $\xi$ for $Z$ in (23) and using (25) we immediately get $\mathcal{L}_V\xi = 4(n + 1)\xi$. Operating (25) by $d$, noting $d$ commutes with $\mathcal{L}_V$ and using the first equation of (2) we find

$$(\mathcal{L}_V d\eta)(X, Y) = -4(n + 1)g(X, \varphi Y)$$

Its comparison with the Lie-derivative of the first equation of (2) and the use of (23) yields $\mathcal{L}_V\varphi = 0$, completing the proof.

Before proving Theorem 2, we state and prove the following lemma.

**Lemma 1** If a vector field $V$ leaves the structure tensor $\varphi$ of the contact metric manifold $M$ invariant, then there exists a constant $c$ such that

(i) $\mathcal{L}_V \eta = c\eta$, (ii) $\mathcal{L}_V \xi = -c\xi$, (iii) $\mathcal{L}_V g = c(g + \eta \otimes \eta)$.

Though this lemma was proved by Mizusawa in [10], to make the paper self-contained, we provide a slightly different proof as follows.

**Proof:** Lie-differentiating the formulas $\varphi \xi = 0$ and $\eta(\varphi X) = 0$ and using $\mathcal{L}_V \varphi = 0$, we find $\mathcal{L}_V \xi = -c\xi$, and $\mathcal{L}_V \eta = c\eta$ for a smooth function $c$ on $M$. Next, Lie-derivative of the formula $\eta(X) = g(X, \xi)$ along $V$ gives

$$(\mathcal{L}_V g)(X, \xi) = 2c\eta(X) \quad (26)$$

The Lie-derivative of the first equation of (2) along $V$ provides

$$(\mathcal{L}_V g)(X, \varphi Y) = ((dc) \wedge \eta)(X, Y) + cg(X, \varphi Y) \quad (27)$$

Substituting $\xi$ for $Y$ in the above equation we get $dc = (\xi c)\eta$. Taking its exterior derivative, and then exterior product with $\eta$ shows $(\xi c)(d\eta) \wedge \eta = 0$. By definition of the contact structure, $(d\eta) \wedge \eta$ is nowhere zero on $M$, and so $\xi c = 0$. Hence $dc = 0$, i.e. $c$ is constant. Using this consequence, and equations (26) and (27) we obtain (iii), completing the proof.

**Proof Of Theorem 2:** By virtue of Lemma 1, we have

$$(\mathcal{L}_V g)(Y, Z) = c\{g(Y, Z) + \eta(Y)\eta(Z)\} \quad (28)$$

Differentiating this and using (3) we get

$$(\nabla_X \mathcal{L}_V g)(Y, Z) = -c\{\eta(Z)g(Y, \varphi X + \varphi h X) + \eta(Y)g(Z, \varphi X + \varphi h X)\} \quad (29)$$
Equation (10) can be written

\[ (\nabla_X \mathcal{L}_V g)(Y, Z) = g((\mathcal{L}_V \nabla)(X, Y), Z) + g((\mathcal{L}_V \nabla)(X, Z), Y) \]  

(30)

A straightforward computation using (29) and (30) shows

\[ (\mathcal{L}_V \nabla)(Y, Z) = -c\{\eta(Z)\varphi Y + \eta(Y)\varphi Z + g(Y, \varphi h Z)\xi\} \]

Its covariant differentiation and use of (2) provides

\[ (\nabla_X \mathcal{L}_V \nabla)(Y, Z) = -c\{\eta(Z)(\nabla_X \varphi)Y + \eta(Y)(\nabla_X \varphi)Z \]

\[ - g(Z, \varphi X + \varphi h X)\varphi Y - g(Y, \varphi X + \varphi h X)\varphi Z \]

\[ - g(\varphi h Y, Z)(\varphi X + \varphi h X) + g((\nabla_X \varphi h) Y, Z)\xi\} \]

Using this in the commutation formula (13) for a Riemannian manifold, contracting at \(X\), and using equations (2), (3) and also the well known formula:

\( (\text{div}\varphi)X = -2n\eta(X) \) for a contact metric (see [1]), we find

\[ (\mathcal{L}_V \text{Ric})(Y, Z) = c\{-2g(Y, Z) + 2g(h Y, Z) \]

\[ + 2(2n + 1)\eta(Y)\eta(Z)\} - cg((\nabla_X \varphi h) Y, Z) \]  

(31)

Also, Lie-differentiating (9) along \(V\) and using Lemma 1 we have

\[ (\mathcal{L}_V \text{Ric})(Y, Z) = (V \alpha + c\alpha)g(Y, Z) + (V \beta + c(\alpha + 2\beta))\eta(Y)\eta(Z) \]

(32)

Comparing the previous two equations shows that

\[ [V \alpha + c(\alpha + 2)]g(Y, Z) + [V \beta + c(\{\alpha + 2\beta - 2(2n + 1)\}]\eta(Y)\eta(Z) \]

\[ -c[2g(h Y, Z) - g((\nabla_X \varphi h) Y, Z)] = 0 \]

On one hand, we substitute \(Y = Z = \xi\) in the above equation getting one equation, and on the other hand, we contract the above equation (noting that both \(h\) and \(\varphi h\) are trace-free) getting another equation. Solving the two equations we obtain

\[ V \alpha + c(\alpha + 2) = 0, \quad V \beta + c(\alpha + 2\beta - 4n - 2) = 0 \]

(33)

The \(g\)-trace of equation (9) gives the scalar curvature

\[ r = (2n + 1)\alpha + \beta \]  

(34)
The divergence of (9) along with the contracted second Bianchi identity yields
\[ dr = 2 d\alpha + 2(\xi \beta) \eta. \]
Taking its exterior derivative, and then exterior product with \( \eta \) we have
\[ (\xi \beta) \eta \wedge d\eta = 0. \]
As \( \eta \wedge d\eta \) vanishes nowhere on \( M \), we find \( \xi \beta = 0 \) whence \( dr = 2 d\alpha \). Hence \( V \alpha = V r = 0 \), by hypothesis. Thus, it follows from (34) that \( V \beta = 0 \). Consequently, equations (33) reduce to:
\[ c(\alpha + 2) = 0 \] and \( c(\alpha + 2 \beta - 4n - 2) = 0 \), and hence imply that, either \( c = 0 \) in which case \( V \) is an infinitesimal automorphism, or \( \alpha = -2 \) and \( \alpha + 2 \beta = 4n+2 \). In the second case, adding the two equations gives \( \alpha + \beta = 2n \).

But, from equation (9) we have \( \alpha + \beta = Tr.l \). Therefore, \( Tr.l = 2n \), and thus, \( V \) is an infinitesimal automorphism, or \( \alpha = -2 \), the \( \eta \)-Einstein structure is \( D \)-homothetically fixed, completing the proof.

4 An Explicit Example

An explicit example of non-trivial Ricci soliton as a Sasakian metric is the \((2n+1)\)-dimensional Heisenberg group \( \mathcal{H}^{2n+1} \) (which arose from quantum mechanics) of matrices of type
\[ \begin{bmatrix} 1 & Y & z \\ O^t & I_0 & X^t \\ 0 & O & 1 \end{bmatrix}, \]
where \( X = (x_1, \ldots, x_n), Y = (y_1, \ldots, y_n), O = (0, \ldots, 0) \in \mathbb{R}^n, z \in \mathbb{R} \). As a manifold, this is just \( \mathbb{R}^{2n+1} \) with coordinates \((x^i, y^i, z)\) where \( i = 1, \ldots, n \), and has the left-invariant Sasakian structure \((\eta, \xi, \varphi, g)\) defined by
\[ \eta = \frac{1}{2} (dz - \sum_{i=1}^n y^i dx^i), \xi = \frac{\partial}{\partial z}, \varphi(\frac{\partial}{\partial x^i}) = -\frac{\partial}{\partial y^i}, \varphi(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}, \varphi(\frac{\partial}{\partial z}) = 0, \] and the Riemannian metric
\[ g = \eta \otimes \eta + \frac{1}{2} \sum_{i=1}^n (dx^i)^2 + (dy^i)^2. \] Its \( \varphi \)-sectional curvature (i.e. the sectional curvature of plane sections orthogonal to \( \xi \)) is equal to \(-3\), so its Ricci tensor satisfies equation (20), as shown by Okumura [11], and hence \( \mathcal{H}^{2n+1} \) is a \( D \)-homothetically fixed null \( \eta \)-Einstein manifold. Setting
\[ V = \sum_{i=1}^n (V^i \frac{\partial}{\partial x^i} + \tilde{V}^i \frac{\partial}{\partial y^i}) + V^z \frac{\partial}{\partial z}, \] using equations: \( \mathcal{L}_V \xi = 4(n+1)\xi \) and \( \mathcal{L}_V \varphi = 0 \) obtained in the proof of Theorem 1, and the aforementioned actions of \( \varphi \) on the coordinate basis vectors, shows that \( V^i \) and \( \tilde{V}^i \) do not depend on \( z \) and yields the PDEs:
\[
\begin{align*}
\frac{\partial V^i}{\partial x^j} &= \frac{\partial V^i}{\partial y^j}, & \frac{\partial V^i}{\partial y^j} &= -\frac{\partial V^i}{\partial x^j}, & y^i \frac{\partial V^i}{\partial y^j} &= \frac{\partial V^z}{\partial y^j}, \\
V^j &= y^j \frac{\partial V^z}{\partial z} - y^i \frac{\partial V^i}{\partial y^j}, & \frac{\partial V^z}{\partial y^j} &= -4(n+1)
\end{align*}
\]
The last equation readily integrates as \( V^z = -4(n + 1)z + F(x^i, y^i) \). For a special solution, assuming \( F = 0 \), \( V^i = cx^i \), \( \bar{V}^i = cy^i \) and substituting in the above PDEs, we get \( c = -2(n + 1) \), and hence the Ricci soliton vector field \( V = -2(n + 1)(x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} + 2z \frac{\partial}{\partial z}) \). For dimension 3, this reduces to \( V = -4(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2z \frac{\partial}{\partial z}) \) which occurs on p. 37 of [4] without the factor 4, but gets adjusted with our \( \lambda = 6 \) which is 4 times their \( \lambda = 3/2 \).

**Remark 4** Another conclusion that we draw for Theorem 1 is the following: The value \(-2n\) for the scalar curvature \( r \) obtained during the proof, and the equation (17) show that the generalized Tanaka-Webster scalar curvature [1] \( W = r - \text{Ric}(\xi, \xi) + 4n \) vanishes.

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